

Zeta Functions, Determinants and Torsion for Open Manifolds

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Abstract

On an open manifold, the spaces of metrics or connections of bounded geometry, respectively, split into an uncountable number of components. We show that for a pair of metrics or connections, belonging to the same component, relative ζ -functions, determinants, torsion for pairs of generalized Dirac operators are well defined.

1 Introduction

Many of the most important invariants which are defined for closed manifolds don't make sense for open manifolds. Integrals defining e. g. characteristic numbers in general diverge. The spectrum of elliptic self-adjoint operators is not purely discrete etc.. One successful approach is to restrict to bounded geometry, to fix one metric in this class and to consider relative invariants. Concerning self-adjoint differential operators associated to geometry, bounded geometry always implies that their spectrum contains a half line, i. e. it is very far from being discrete. Hence ζ -functions don't make sense. But if we fix a component $comp(g_0)$ in the completed space $\mathcal{M}^{p,r}(I, B_k)$ of metrics g satisfying the conditions

$$(I) \quad r_{inj}(M, g) = \inf_{x \in M} r_{inj}(x) > 0 ,$$

$$(B_k) \quad |(\nabla^g)^i R^g| \leq C_i, \quad 0 \leq i \leq k ,$$

then we can consider for instance the pair $\Delta_q(g), \Delta_q(g_0), g \in comp(g_0)$, Δ_q the Laplace operator acting on q -forms. Then, for $k \geq r > n + 2, p = 1$, we show that

$$e^{-t\Delta_q(g)} - e^{-t\Delta_q(g_0)} \quad (1.1)$$

and

$$\Delta_q(g)e^{-t\Delta_q(g)} - \Delta_q(g_0)e^{-t\Delta_q(g_0)} \quad (1.2)$$

are for $t > 0$ of trace class, and their trace norms are uniformly bounded on compact t -intervals.

This is a consequence of our extended approach to generalized Dirac operators, considering the completed space $C_E^{p,r}(B_k)$ of Clifford connections.

Let $g' \in \text{comp}(g) \subset \mathcal{M}^{1,r}(I, B_k)$, $\nabla' \in \text{comp}(\nabla) \subset C_E^{1,r}(B_k)$, $k \geq r > n + 2$, $D = D(g, \nabla)$, $D' = D'(g', \nabla')$ be the generalized Dirac operators, then

$$e^{-tD'^2} - e^{-tD^2}, \quad (1.3)$$

$$D'e^{-tD'^2} - De^{-tD^2} \quad (1.4)$$

are of trace class and their trace norms are uniformly bounded on compact t -intervals. Assuming additionally $\inf \sigma_e(D^2) > 0$, we define relative ζ -functions, determinants and torsion in the case of the Laplace operator.

The paper is organized as follows. In section 2 we present the necessary facts concerning Clifford bundles, generalized Dirac operators and Sobolev spaces. Section 3 is devoted to spaces of metrics and connections. Section 4 contains some general heat kernel estimates which are needed in section 5. We present in section 5 the first essential step of our approach, proving that for fixed g and variation of the Clifford connection ∇ to ∇' the operators $e^{-tD^2} - e^{-tD'^2}$, $De^{-tD^2} - D'e^{-tD'^2}$ are of trace class and their trace norm is uniformly bounded on compact t -intervals $[a_0, a_1]$, $a_0 > 0$. Section 6 is devoted to the generalization of 5, admitting variation of the bundle metric and the Clifford structure too. We apply our results in sections 7 and 8, establishing certain relative index theorems and defining ζ -functions, determinants and torsion. In a forthcoming paper we drop the assumption $\inf \sigma_e(D^2) > 0$, define relative η -functions and present further applications.

2 Clifford bundles, generalized Dirac operators and Sobolev spaces

We recall for completeness very shortly the basic properties of generalized Dirac operators on open manifolds. Let (M^n, g) be a Riemannian manifold, $m \in M$, $Cl(T_m M, g_m)$ the corresponding Clifford algebra at m . $Cl(T_m M, g_m)$ shall be complexified or not, depending on the other bundles and structure under consideration. A hermitian vector bundle $E \rightarrow M$ is called a bundle of Clifford modules if each fibre E_m is a Clifford module over $Cl(T_m M, g_m)$ with skew symmetric Clifford multiplication. We assume E to be endowed with a compatible connection ∇^E , i.e. ∇^E is metric and

$$\nabla_X^E(Y \cdot \Phi) = (\nabla_X^g Y) \cdot \Phi + Y \cdot (\nabla_X^E \Phi),$$

$X, Y \in \Gamma(TM)$, $\Phi \in \Gamma(E)$. Then we call the pair (E, ∇^E) a Clifford bundle. The composition

$$\Gamma(E) \xrightarrow{\nabla} \Gamma(T^*M \otimes E) \xrightarrow{g} \Gamma(TM \otimes E) \longrightarrow \Gamma(E)$$

shall be called the generalized Dirac operator D . We have $D = D(g, E, \nabla)$. If X_1, \dots, X_n is an orthonormal basis in $T_m M$ then

$$D = \sum_{i=1}^n X_i \cdot \nabla_{X_i}^E.$$

D is of first order elliptic, formally self-adjoint and

$$D^2 = \Delta^E + \mathcal{R},$$

where $\Delta^E = (\nabla^E)^* \nabla^E$ and $\mathcal{R} \in \Gamma(\text{End}(E))$ is the bundle endomorphism

$$\mathcal{R}\Phi = \frac{1}{2} \sum_{i,j=1}^n X_i X_j R^E(X_i, X_j) \Phi.$$

Next we recall some associated functional spaces and their properties if we assume bounded geometry. These facts are contained in [5], [7], [2].

Let $E \rightarrow M$ be a Clifford bundle, $\nabla = \nabla^E$, D the generalized Dirac operator. Then we define for $\Phi \in \Gamma(E)$, $p \geq 1$, $r \in \mathbf{Z}$, $r \geq 0$,

$$\begin{aligned} |\Phi|_{W^{p,r}} &:= \left(\int \sum_{i=0}^r |\nabla^i \Phi|_x^p d\text{vol}_x(g) \right)^{\frac{1}{p}}, \\ |\Phi|_{H^{p,r}} &:= \left(\int \sum_{i=0}^r |D^i \Phi|_x^p d\text{vol}_x(g) \right)^{\frac{1}{p}}, \\ W_r^p(E) &:= \{ \Phi \in \Gamma(E) \mid |\Phi|_{W^{p,r}} < \infty \}, \\ W^{p,r}(E) &:= \text{completion of } W_r^p \text{ w. r. t. } | \cdot |_{W^{p,r}}, \\ H_r^p(E) &:= \{ \Phi \in \Gamma(E) \mid |\Phi|_{H^{p,r}} < \infty \}, \\ H^{p,r}(E) &:= \text{completion of } H_r^p \text{ w. r. t. } | \cdot |_{H^{p,r}}. \end{aligned}$$

In a great part of our consideration we restrict to $p = 2$. In this case we write $W^{2,r} \equiv W^r$, $H^{2,r} \equiv H^r$ etc.. If $r < 0$ then we set

$$\begin{aligned} W^r(E) &:= \left(W^{-r}(E) \right)^*, \\ H^r(E) &:= \left(H^{-r}(E) \right)^*. \end{aligned}$$

Assume (M^n, g) complete. Then $C_c^\infty(E)$ is a dense subset of $W^{p,1}(E)$ and $H^{p,1}(E)$. This follows from proposition 1.4 in [2]. If we use this density and the fact

$$|D\Phi(m)| \leq C \cdot |\nabla\Phi(m)|,$$

we obtain $|\Phi|_{H^{p,1}} \leq C' \cdot |\Phi|_{W^{p,1}}$ and a continuous embedding

$$W^{p,1}(E) \hookrightarrow H^{p,1}(E).$$

For $r > 1$ this cannot be established, and we need further assumptions. Consider as in the introduction the following conditions

$$\begin{aligned} (I) \quad & r_{inj}(M, g) = \inf_{x \in M} r_{inj}(x) > 0, \\ (B_k(M, g)) \quad & |(\nabla^g)^i R^g| \leq C_i, \quad 0 \leq i \leq k, \\ (B_k(E, \nabla^E)) \quad & |(\nabla^g)^i R^E| \leq C_i, \quad 0 \leq i \leq k. \end{aligned}$$

It is a well known fact that for any open manifold and given $k, 0 \leq k \leq \infty$, there exists a metric g satisfying (I) and $(B_k(M, g))$. Moreover, (I) implies completeness of g .

Lemma 2.1. *Assume (M^n, g) with (I) and (B_k) . Then $C_c^\infty(E)$ is a dense subset of $W^{p,r}(E)$ and $H^{p,r}(E)$ for $0 \leq r \leq k + 2$.*

See [5], proposition 1.6 for a proof. □

Lemma 2.2. *Assume (M^n, g) with (I) and (B_k) . Then there exists a continuous embedding*

$$W^{p,r}(E) \hookrightarrow H^{p,r}(E), \quad 0 \leq r \leq k + 1.$$

Proof. According to 2.1, we are done if we could prove

$$|\Phi|_{H^{p,r}} \leq C \cdot |\Phi|_{W^{p,r}}$$

for $0 \leq r \leq k + 1$ and $\Phi \in C_c^\infty(E)$. Perform induction. For $r = 0$, $|\Phi|_{H^{p,0}} = |\Phi|_{W^{p,0}}$. Assume $|\Phi|_{H^{p,r}} \leq C \cdot |\Phi|_{W^{p,r}}$. Then

$$\begin{aligned} |\Phi|_{H^{p,r+1}} &\leq C \cdot (|\Phi|_{H^{p,r}} + |D\Phi|_{H^{p,r}}) \\ &\leq C \cdot (|\Phi|_{W^{p,r}} + |D\Phi|_{W^{p,r}}). \end{aligned}$$

Let $\frac{\partial}{\partial x^i}, i = 1, \dots, n$ be coordinate vectors fields which are orthonormal in $m \in M$. Then with $\nabla_i = \nabla \frac{\partial}{\partial x^i}$

$$|\nabla^s D\Phi|_m^p \leq C \cdot \sum_{i_1, \dots, i_s, j} |\nabla_{i_1} \dots \nabla_{i_s} \frac{\partial}{\partial x^j} \cdot \nabla_j \Phi|^p.$$

Now we apply the Leibniz rule and use the fact that in an atlas of normal charts the Christoffel symbols have bounded euclidean derivatives up to order $k - 1$. This yields

$$|\nabla^r D\Phi|_m^p \leq C \cdot \sum_{i_1, \dots, i_{r+1}} |\nabla_{i_1} \dots \nabla_{i_{r+1}} \Phi|_m^p \quad \text{for } r \leq k,$$

i. e.

$$|D\Phi|_{W^{p,r}} \leq C \cdot |\Phi|_{W^{p,r+1}}$$

altogether

$$|\Phi|_{H^{p,r+1}} \leq C \cdot |\Phi|_{W^{p,r+1}}.$$

□

Remark. For $p = 2$ this proof is contained in [2].

□

Theorem 2.3. Assume (M^n, g) with (I) and (B_k) and (E, ∇) with (B_k) and $p = 2$. Then for $r \leq k$

$$H^{2,r}(E) \equiv H^r(E) \cong W^r(E) \equiv W^{2,r}(E)$$

as equivalent Hilbert spaces.

Proof. According to 2.2., $W^r(E) \subseteq H^r(E)$ continuously. Hence we have to show $H^r(E) \subseteq W^r(E)$ continuously. The latter follows from the local elliptic inequality, a uniformly locally finite cover by normal charts of fixed radius, trivializations and the existence of elliptic constants. The proof is performed in [2].

□

Remark. 2.3 holds for $1 < p < \infty$ (cf. [13]).

□

As it is clear from the definition, the spaces $W^{p,k}(E)$ can be defined for any Riemannian vector bundle (E, h_E, ∇^E) . We assume this more general case and define additionally

$${}^{b,s}W(E) := \left\{ \varrho \in C^S(E) \mid {}^{b,s}|\varrho| := \sum_{i=0}^s \sup_{x \in M} |\nabla^i \varrho|_x < \infty \right\}$$

and in the case of a Clifford bundle

$${}^{b,s}H(E) := \left\{ \varrho \in C^S(E) \mid {}^{b,s,D}|\varrho| := \sum_{i=0}^s \sup_{x \in M} |D^i \varrho|_x < \infty \right\}.$$

${}^{b,s}W(E)$ is a Banach space and coincides with the completion of the space of all $\varrho \in \Gamma(E)$ with ${}^{b,s}|\varrho| < \infty$ with respect to ${}^{b,s}|\cdot|$.

Theorem 2.4. Let (E, h, ∇^E) be a Riemannian vector bundle satisfying (I), $(B_k(M^n, g))$, $B_k(E; \nabla)$.

a. Assume $k \geq r, k \geq 1, r - \frac{n}{p} \geq s - \frac{n}{q}, r \geq s, q \geq p$, then

$$W^{p,r}(E) \hookrightarrow W^{q,s}(E) \tag{2.1}$$

continuously.

b. If $k \geq 0, r > \frac{n}{p} + s$ then

$$W^{p,r}(E) \hookrightarrow {}^{q,s}W(E) \tag{2.2}$$

continuously.

We refer to [7] for the proof.

□

Corollary 2.5. Let $E \rightarrow M$ be a Clifford bundle satisfying (I), $(B_k(M))$, $(B_k(E))$, $k > r > \frac{n}{2} + s$. Then

$$H^r(E) \hookrightarrow {}^{b,s}H(E) \tag{2.3}$$

continuously.

Proof. We apply 2.3, (2.2) and obtain

$$H^r(E) \hookrightarrow {}^{b,s}W(E). \quad (2.4)$$

Quite similar as in the proof of 2.2.,

$$H^r(E) \hookrightarrow {}^{b,s}W(E). \quad (2.5)$$

continuously. \square

A key role for anything in the sequel plays the module structure theorem for Sobolev spaces.

Theorem 2.6. *Let $(E_i, h_i, D_i) \rightarrow (M^n, g)$ be vector bundles with $(I), (B_k(M^n, g)), (B_k(E_i, \nabla_i)), i = 1, 2$. Assume $0 \leq r \leq r_1, r_2 \leq k$. If $r = 0$ assume*

$$\left\{ \begin{array}{l} r - \frac{n}{p} < r_1 - \frac{n}{p_1} \\ r - \frac{n}{p} < r_2 - \frac{n}{p_2} \\ r - \frac{n}{p} \leq r_1 - \frac{n}{p_1} + r_2 - \frac{n}{p_2} \\ \frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2} \end{array} \right\} \text{ or } \left\{ \begin{array}{l} r - \frac{n}{p} \leq r_1 - \frac{n}{p_1} \\ 0 < r_2 - \frac{n}{p_2} \\ \frac{1}{p} \leq \frac{1}{p_1} \end{array} \right\} \text{ or } \left\{ \begin{array}{l} 0 < r_1 - \frac{n}{p_1} \\ r - \frac{n}{p} \leq r_2 - \frac{n}{p_2} \\ \frac{1}{p} \leq \frac{1}{p_2} \end{array} \right\}. \quad (2.6)$$

If $r > 0$ assume $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}$ and

$$\left\{ \begin{array}{l} r - \frac{n}{p} < r_1 - \frac{n}{p_1} \\ r - \frac{n}{p} < r_2 - \frac{n}{p_2} \\ r - \frac{n}{p} \leq r_1 - \frac{n}{p_1} + r_2 - \frac{n}{p_2} \end{array} \right\} \text{ or } \left\{ \begin{array}{l} r - \frac{n}{p} \leq r_1 - \frac{n}{p_1} \\ r - \frac{n}{p} \leq r_2 - \frac{n}{p_2} \\ r - \frac{n}{p} < r_1 - \frac{n}{p_1} + r_2 - \frac{n}{p_2} \end{array} \right\}. \quad (2.7)$$

Then the tensor product of sections defines a continuous bilinear map

$$W^{p_1, r_1}(E_1, \nabla_1) \times W^{p_2, r_2}(E_2, \nabla_2) \longrightarrow W^{p, r}(E_1 \otimes E_2, \nabla_1 \otimes \nabla_2).$$

We refer to [7] for the proof. \square

Define for $u \in C^0(M), c > 0$

$$\bar{u}_c(x) := \frac{1}{\text{vol} B_c(x)} \int_{B_c(x)} u(y) d\text{vol}_y(g).$$

Lemma 2.7. *Let (M^n, g) be complete $\text{Ric}(g) \geq k, k \in \mathbf{R}$. Then there exists a positive constant $C = C(n, k, R)$, depending only on n, k, R such that for any $c \in]0, R[$ and any $u \in W^{1,1}(M) \cap C^\infty(M)$*

$$\int_M |u - \bar{u}_c| d\text{vol}_x(g) \leq C \cdot c \cdot \int_M |\nabla u| d\text{vol}_x(g).$$

Proof. For $u \in C_c^\infty(M)$ the proof is performed in [10], p. 31–33. But what is only needed in the proof is $\int |u|dx, \int |\nabla u|dx < \infty$ (even only $\int |\nabla u|dx < \infty$).

The key is the lemma of Buser,

$$\int_{B_c(x)} |u - \bar{u}_c|dx \leq C \cdot c \cdot \int_{B_c(x)} |\nabla u|dx.$$

□

Remark. Even $u, \nabla u \in C^\infty$ are not necessary, $u \in C^1$ is completely sufficient.

□

Proposition 2.8. *Let $(E, h, \nabla) \rightarrow (M^n, g)$ be a Riemannian vector bundle, (M^n, g) with (I) , $(B_0), r > n + 1, 0 < c < r_{inj}$ and $\eta \in W^{1,r}(E)$. Then $\overline{|\eta|}_c \in W^{1,0}(M) \equiv L_1(M)$, where*

$$\overline{|\eta|}_c(x) := \frac{1}{\text{vol} B_c(x)} \int_{B_c(x)} |\eta(y)|dy.$$

Proof. Set $u(x) = |\eta(x)|$. Then $\overline{u}_c(x) = \overline{|\eta|}_c(x)$ and, according to Kato's inequality,

$$\int |\nabla u|dx = \int |\nabla |\eta||dx \leq \int |\nabla \eta|dx < \infty.$$

Hence we obtain from 2.7, $|u| = |\eta| \in L_1, |\eta| - \overline{|\eta|}_c \in L_1$,

$$\overline{|\eta|}_c \in L_1. \tag{2.8}$$

□

Remark. For (2.2) is the assumption $(B_0(E))$ superfluous. Nevertheless, we have in our applications even $(B_k(E))$. □

Finally we recall for clarity and distinctness a fact which will be very important later. Let $(E, h, \nabla) \rightarrow (M^n, g)$ be a Riemannian vector bundle with $(I), (B_k(M)), (B_k(E)), k \geq r + 1, r > \frac{n}{p} + 1, 0 < c < r_{inj}$. Then the spaces $W^{p,r}(E|_{B_c(x)}) = \{\varrho | \varrho \text{ distributional section of } E|_{B_c(x)} \text{ s. t. } |\varrho|_{p,r} < \infty\}$ are well defined, $x \in M$ arbitrary. Radial parallel translation of an orthonormal basis defines an isomorphism

$$A_x : W^{p,r}(E|_{B_c(x)}) \xrightarrow{\cong} W^{p,r}(B_c(0), V^N), \tag{2.9}$$

$B_c(0) \subset \mathbf{R}^n, V^N = \mathbf{R}^N$ or $\mathbf{C}^N, N = rkE$. We conclude from $(B_k(M)), (B_k(E)), k \geq r + 1$ and [7] that there exists constants c_1, C_1 s. t.

$$c_1 \cdot |\varrho|_{p,r,B_c(x)} \leq |A_x \varrho|_{p,r,B_c(0)} \leq C_1 \cdot |\varrho|_{p,r,B_c(x)}, \tag{2.10}$$

c_1, C_1 independent of x . Moreover, if $\varrho \in W^{p,r}(E)$ then $\varrho|_{B_c(x)} \in W^{p,r}(E|_{B_c(x)})$. Similarly,

$$c_2^{b,s} |\varrho|_{B_c(x)} \leq {}^{b,s} |A_x \varrho|_{B_c(0)} \leq C_2 \cdot {}^{b,s} |\varrho|_{B_c(x)}, \quad (2.11)$$

c_2, C_2 independent of x . $(B_k(M)), (B_k(E)), 0 < c < r_{inj}$ imply that $B_c(x)$ satisfies all required smoothness conditions and we obtain from the Sobolev embedding theorem, (2.10), (2.11)

$$W^{p,r}(E|_{B_c(x)}) \hookrightarrow {}^{b,1}W(E|_{B_c(x)}), \quad (2.12)$$

$${}^{b,1} |\varrho|_{B_c(x)} \leq C \cdot |\varrho|_{p,r,B_c(x)}, \quad (2.13)$$

C independent of x .

3 Uniform spaces of metrics and connections

Denote by $\mathcal{M}(I, B_k)$ the set of all metrics g satisfying the conditions (I) and (B_k) .

Let $1 \leq p < \infty, k \geq r \geq \frac{n}{p} + 2, \delta > 0$ and set

$$\begin{aligned} V_\delta &= \left\{ (g, g') \in \mathcal{M}(I, B_k)^2 \mid g \text{ and } g' \text{ are quasi isometric and } |g - g'|_{g,p,r} \right. \\ &\quad \left. := \left(\int \left(|g - g'|_{g,x}^p + \sum_{i=0}^{r-1} |(\nabla^g)^i (\nabla^g - \nabla^{g'})|_{g,x}^p \right) d\text{vol}_x(g) \right)^{\frac{1}{p}} < \delta \right\}. \end{aligned}$$

Here g, g' quasi isometric means $C_1 \cdot g \leq g' \leq C_2 \cdot g$ in the sense of quadratic forms. This is equivalent to ${}^b |g - g'|_g < \infty$ and ${}^b |g - g'|_{g'} < \infty$, where for a tensor t ${}^b |t|_g = \sup_{x \in M} |t|_{g,x}$.

Proposition 3.1. *Assume p, k, r as above. Then $\mathcal{B} = \{V_\delta\}_{\delta > 0}$ is a basis for a metrizable uniform structure $\mathcal{U}^{p,r}(\mathcal{M}(I, B_k))$.*

We refer to [6] for a proof. The key to the proof is the module structure theorem.

□

Let $\mathcal{M}_r^p(I, B_k) = \mathcal{M}(I, B_k)$ endowed with the topology. $\mathcal{M}^{p,r} := \overline{\mathcal{M}_r^p}$ the completion. If $k \geq r > \frac{n}{p} + 1$ then $\mathcal{M}^{p,r}$ still consists of C^1 -metrics, i.e. does not contain semi definite elements. This has been proved by Salomonsen in [12].

Theorem 3.2. *Let $k \geq r > \frac{n}{p} + 2, g \in \mathcal{M}(I, B_k)$, $U^{p,r}(g) = \{g' \in \mathcal{M}^{p,r}(I, B_k) \mid {}^b |g - g'|_g < \infty, {}^b |g - g'|_{g'} < \infty \text{ and } |g - g'|_{g,p,r} < \infty\}$ and denote by $\text{comp}(g) \subset \mathcal{M}^{p,r}(I, B_k)$ the component of g in $\mathcal{M}^{p,r}(I, B_k)$. Then*

$$\text{comp}(g) = U^{p,r}(g), \quad (3.1)$$

$\text{comp}(g)$ is a Banach manifold, for $p = 2$ a Hilbert manifold and $\mathcal{M}^{p,r}(I, B_k)$ has a representation as topological sum

$$\mathcal{M}^{p,r}(I, B_k) = \sum_{j \in J} \text{comp}(g_j) \quad (3.2)$$

J an uncountable set.

The proof is performed in [6]. □

Remarks.

1. If M^n is compact then J consists of one element.
2. All metrics in the completed space are at least of class C^2 . Hence curvature is well defined. □

Let $(E, h) \rightarrow (M^n, g)$ be a Clifford bundle without a fixed connection, (M^n, g) with (I) and (B_k) .

Set

$$C_E(B_k) = \left\{ \nabla \mid \begin{array}{l} \text{is Clifford connection, metric with respect to } h \text{ and satisfies} \\ (B_k(E, \nabla)) \end{array} \right\}$$

Assume $(E, h) \rightarrow (M^n, g)$ as above, $k \geq r > \frac{n}{p} + 2, \delta > 0$ and set

$$\begin{aligned} V_\delta &= \left\{ (\nabla, \nabla') \in C_E(B_k)^2 \mid |\nabla - \nabla'|_{\nabla, p, r} \right. \\ &\quad \left. := \left(\int \sum_{i=0}^r |\nabla(\nabla - \nabla')|_x^p d\text{vol}_x(g) \right)^{\frac{1}{p}} < \delta \right\}. \end{aligned}$$

Proposition 3.3. *Assume p, k, r as above. Then $\mathcal{B} = \{V_\delta\}_{\delta > 0}$ is a basis for a metrizable structure $\mathcal{U}^{p, r}(C_E(B_k))$.*

We refer to [4] for a proof. □

Let $C_E^{p, r}(B_k)$ be the completion of $(C_E(B_k), \mathcal{U}^{p, r}(C_E(B_k)))$. If $\nabla, \nabla' \in C_E^{p, r}(B_k)$ then $\nabla - \nabla'$ is a 1-form η with values in $\mathcal{G}_E = \text{skew endomorphisms}$ satisfying

$$\eta_x(Y \cdot \Phi) = Y \cdot \eta_x(\Phi). \quad (3.3)$$

As well known, a metric connection ∇ in E induces a connection ∇ in \mathcal{G}_E . Denote

$$\begin{aligned} \Omega^1(\mathcal{G}_E^{Cl}) &:= \{ \eta \in \Omega^1(\mathcal{G}_E) \mid \eta \text{ satisfies (3.3)} \}, \\ \Omega_r^{1, p}(\mathcal{G}_E^{Cl}, \nabla) &:= \left\{ \eta \in \Omega^1(\mathcal{G}_E^{Cl}) \mid \right. \\ &\quad \left. |\eta|_{\nabla, p, r} := \left(\int \sum_{i=0}^r |\nabla^i \eta|_x^p d\text{vol}_x(g) \right)^{\frac{1}{p}} < \infty \right\}, \\ \Omega^{1, p, r}(\mathcal{G}_E^{Cl}, \nabla) &:= \overline{\Omega_r^{1, p}(\mathcal{G}_E^{Cl}, \nabla)}^{\|\cdot\|_{\nabla, p, r}}. \end{aligned}$$

If (M^n, g) satisfies $(I), (B_k)$ then

$$\Omega^{1, p, r}(\mathcal{G}_E^{Cl}, \nabla) := \overline{C_c^\infty(\mathcal{G}_E)}^{\|\cdot\|_{\nabla, p, r}} = \left\{ \eta \text{ distributional} \mid |\eta|_{\nabla, p, r} < \infty \right\}, \quad (3.4)$$

$$r \leq k + 2.$$

Theorem 3.4. *Assume $(E, h) \rightarrow (M^n, g), p, k, r$ as above. Denote for $\nabla \in C_E(B_k)$ by $\text{comp}(\nabla) \subset C_E^{p, r}(B_k)$ the component of ∇ in $C_E^{p, r}(B_k)$. Then*

$$\text{comp}(\nabla) = \nabla + \Omega^{1, p, r}(\mathcal{G}_E^{Cl}, \nabla) \quad (3.5)$$

and $C_E^{p,r}(B_k)$ has a representation as topological sum

$$C_E^{p,r}(B_k) = \sum_{j \in J} \text{comp}(\nabla_j). \quad (3.6)$$

The proof is performed in [4]. □

Remarks.

1. If M^n is compact then $C_E^{p,r}(B_k) = C_E^{p,r}$ consists of one component
2. If ∇ is not smooth then one sets $\nabla^i = (\nabla_0 + (\nabla - \nabla_0))^i$, $\nabla_0 \in \text{comp}\nabla \cap C_E(B_k)$, and the right hand side makes sense.
3. All connections in the complete space are at least of class C^2 . Hence curvature is well defined.

For the sequel, we must sharpen our considerations concerning Sobolev spaces. Let $(E, h, \nabla) \rightarrow (M^n, g)$ be a Riemannian vector bundle. The connection ∇ enters into the definition of the Sobolev spaces $W^{p,r}$. Hence we should write $W^{p,r}(E, \nabla)$. Now there arises the natural question, how do the spaces $W^{p,r}(E, \nabla)$ on ∇ ? We present here one answer. Other considerations are performed in [4], [3].

Proposition 3.5. *Let $(E, h, \nabla) \rightarrow (M^n, g)$ be a Riemannian vector bundle with (I) , (B_k) , $(B_k(E, \nabla))$, $k \geq r > \frac{n}{p} + 1$, $1 \leq p < \infty$. Suppose $\nabla' \in \text{comp}(\nabla) \subset C_E^{p,r}(B_k)$, ∇' smooth, i. e. $\nabla' = \nabla + \eta$, $\eta \in \Omega^{1,p,r}(\mathcal{G}_E, \nabla) \cap C^\infty$. Then*

$$W^{p,i}(E, \nabla) = W^{p,i}(E, \nabla'), \quad 0 \leq i \leq r, \quad (3.7)$$

as equivalent Banach spaces.

For the proof we refer to [4], [3]. The proof includes some combinatorial considerations and essentially uses the module structure theorem. This is the reason why we assumed $k \geq r > \frac{n}{p} + 1$. But this assumption can be weakened. We only need the validity of the module structure theorem. □

Remark. The assumption η smooth is superfluous. As we mentioned already several times, we can define $W^{p,r}(E, \nabla')$ and prove (3.7) for $\nabla' \in \text{comp}(\nabla)$ only. □

Corollary 3.6. *Suppose $(E, h, \nabla) \rightarrow (M^n, g)$ as above, $k \geq r > \frac{n}{p} + 2$, $\nabla' = \nabla + \eta$, $\eta \in \Omega^{1,p,r}(\mathcal{G}_E, \nabla)$. Then*

$$W^{2p,i}(E, \nabla) = W^{2p,i}(E, \nabla'), \quad 0 \leq i \leq \frac{r}{2}. \quad (3.8)$$

Proof. $r > \frac{n}{p} + 2$ implies $r - \frac{n}{p} \geq \frac{r}{2} - \frac{n}{2p}$, $2p \geq p$, $r \geq \frac{r}{2}$, i. e.

$$\Omega^{1,p,r}(\mathcal{G}_E, \nabla) \subseteq \Omega^{1,2p,\frac{r}{2}}(\mathcal{G}_E, \nabla).$$

Now we apply 3.5 replacing $p \rightarrow 2p$, $r \rightarrow \frac{r}{2}$. □

Corollary 3.7. Suppose $(E, h, \nabla) \rightarrow (M^n, g)$ a Clifford bundle with the conditions above for $p = 1$, i. e. $k \geq r > n + 2$, $\nabla' \in \text{comp}(\nabla) \subset C_E^{1,r}(B_k)$, ∇' smooth. Then

$$W^{2,i}(E, \nabla) \equiv W^i(E, \nabla) = W^i(E, \nabla') \equiv W^{2,i}(E, \nabla'), \quad 0 \leq i \leq \frac{r}{2}. \quad (3.9)$$

Corollary 3.8. Assume the hypotheses of 3.7. and write $D = D(\nabla, g)$, $D' = D(\nabla', g)$. Then

$$H^i(E, D) = H^i(E, D'), \quad 0 \leq i \leq \frac{r}{2} \quad (3.10)$$

In particular,

$$\mathcal{D}_{D^i} = \mathcal{D}_{D'^i}, \quad 0 \leq i \leq \frac{r}{2} \quad (3.11)$$

where \mathcal{D}_{D^i} denotes the domain of definition of $\overline{D^i}$.

Proof. (3.11) follows from the result of Chernoff that D^i is essentially self adjoint on $C_c^\infty(E)$, $\mathcal{D}_{D^i} = H^i(E, D)$ and (3.10). (3.10) follows from (3.9) and 2.3. \square

Finally we make some remarks concerning the essential spectrum of D and D^2 . More precisely, we prove that it is an invariant of $\text{comp}(\nabla)$. We have several distinct proofs for this and present here a particularly simple one.

We consider Weyl sequences and restrict to orthonormal ones. Denote by $\sigma_e(D)$ the essential spectrum of D . $\lambda \in \sigma_e(D)$ if and only if there exists a Weyl sequence for λ , i. e. an orthonormal sequence $(\Phi_\nu)_\nu$, $\Phi_\nu \in \mathcal{D}_D$; s. t.

$$\lim_{\nu \rightarrow \infty} (D - \lambda)\Phi_\nu = 0. \quad (3.12)$$

Lemma 3.9. Suppose $\lambda \in \sigma_e(D)$. Then there exists a Weyl sequence $(\Phi_\nu)_\nu$ for λ s. t. for any compact subset $K \subset M$

$$\lim_{\nu \rightarrow \infty} |\Phi_\nu|_{L_2(K, E)} = 0. \quad (3.13)$$

This is Lemma 4.29 of [2]. One simply chooses an exhaustion $K_1 \subset K_2 \subset \dots, \bigcup K_i = M$, starts with an arbitrary Weyl sequence $(\Psi_\nu)_\nu$, produces by the Rellich lemma and a diagonal choice a subsequence χ_ν such that $(\chi_\nu)_\nu$ converges on any K_i in the L_2 -sense and defines $\Phi_\nu := (\chi_{2\nu+1} - \chi_{2\nu})/\sqrt{2}$. $(\Phi_\nu)_\nu$ has the desired properties. \square

Proposition 3.10. Suppose $(E, h, \nabla) \rightarrow (M^n, g)$ a Clifford bundle with (I) , $(B_k(M))$, $(B_k(E, \nabla))$, $k \geq r > n + 2, n \geq 2$, $\nabla' \in \text{comp}(\nabla) \subset C_E^{1,r}(B_k)$, $D = D(\nabla, g)$, $D' = D(\nabla', g)$. Then

$$\sigma_e(D) = \sigma_e(D'). \quad (3.14)$$

Proof.

$$D' = \sum_i e_i \nabla'_{e_i} = \sum_i e_i \cdot (\nabla_{e_i} + \eta_{e_i}(\cdot)) = D + \eta^{op},$$

where the operator η^{op} acts as

$$\eta^{op}(\Phi)|_x = \sum_i e_i \cdot \eta_{e_i}(\Phi)|_x.$$

Then, pointwise, $|\eta^{op}|_x \leq C \cdot |\eta|_x$, C independent of x . Given $\varepsilon > 0$, there exists a compact set $K = K(\varepsilon) \subset M$ such that

$$\sup_{x \in M \setminus K} |\eta|_x < \frac{\varepsilon}{C}, \quad i.e. \quad \sup_{x \in M \setminus K} |\eta^{op}|_x < \varepsilon. \quad (3.15)$$

Assume now $\lambda \in \sigma_e(D)$, $(\Phi_\nu)_\nu$ a Weyl sequence as in (3.13). According to (3.11), $\Phi_\nu \in \mathcal{D}_{D'}$. Then

$$(D' - \lambda)\Phi_\nu = (D'_D)\Phi_\nu + (D - \lambda)\Phi_\nu.$$

By assumption, $(D - \lambda)\Phi_\nu \rightarrow 0$. Moreover,

$$|(D' - D)\Phi_\nu|_{L_2(M, E)} = |\eta^{op}\Phi_\nu|_{L_2(M, E)} \leq C \cdot (|\eta\Phi_\nu|_{L_2(K, E)} + |\eta\Phi_\nu|_{L_2(M \setminus K, E)}).$$

$|\eta\Phi_\nu|_{L_2(K, E)} \rightarrow 0$ and

$$C \cdot |\eta\Phi_\nu|_{L_2(M \setminus K, E)} \leq C \cdot \sup_{x \in M \setminus K} |\eta|_x \cdot |\Phi_\nu|_{L_2(M \setminus K, E)} < \varepsilon.$$

Hence $(D' - \lambda)\Phi_\nu \rightarrow 0$, $\lambda \in \sigma_e(D')$, $\sigma_e(D) \subseteq \sigma_e(D')$. Exchanging the role of D, D' , we obtain $\sigma(D') \subseteq \sigma(D)$. \square

4 General heat kernel estimates

We collect some standard facts concerning the heat kernel of e^{-tD^2} . The best references for this are [1], [2].

We consider the self-adjoint closure of D in $L_2(E) = H^0(E)$, $D = \int_{-\infty}^{+\infty} \lambda E_\lambda$.

Lemma 4.1. $\{e^{itD}\}_{t \in \mathbf{R}}$ defines a unitary group on the spaces $H^r(E)$, for $0 \leq h \leq r$ holds

$$|D^h e^{itD}\Psi|_{L_2} = |e^{itD}D^h\Psi|_{L_2} = |D^h\Psi|_{L_2}. \quad (4.1)$$

\square

We can extend this action to $H^{-r}(E)$ by means of duality.

Lemma 4.2. e^{-tD^2} maps $L_2(E) \equiv H^0(E) \rightarrow H^r(E)$ for any $r > 0$ and

$$|e^{-tD^2}|_{L_2 \rightarrow H^r} \leq C \cdot t^{-\frac{r}{2}}, \quad t \in]0, \infty[, \quad C = C(r). \quad (4.2)$$

Proof. Insert into $e^{-tD^2} = \int e^{-t\lambda^2} dE_\lambda$ the equation

$$e^{-t\lambda^2} = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{i\lambda s} e^{-\frac{s^2}{4t}} ds$$

and use

$$\sup |\lambda^r e^{-t\lambda^2}| \leq C \cdot t^{-\frac{r}{2}}.$$

□

Corollary 4.3. *Let $r, s \in \mathbf{Z}$ be arbitrary. Then for $t > 0$ $e^{-tD^2} : H^r(E) \rightarrow H^s(E)$ continuously.*

Proof. This follows from 4.2., duality and the semi group property of $\{e^{-tD^2}\}_{t>0}$. □

e^{-tD^2} has a Schwartz kernel $W \in \Gamma(\mathbf{R}_+ \times M \times M, E \boxtimes E)$,

$$W(t, m, p) = \langle \delta(m), e^{-tD^2} \delta(p) \rangle,$$

where $\delta(m) \in H^{-r}(E) \otimes E_m$ is the map $\Psi \in H^r(E) \rightarrow \langle \delta(m), \Psi \rangle = \Psi(m)$, $r > \frac{n}{2}$. The main result of this section is the fact that for $t > 0$, $W(t, m, p)$ is a smooth integral kernel in L_2 with good decay properties if we assume bounded geometry.

Denote by $C(m)$ the best local Sobolev constant of the map $\Psi \rightarrow \Psi(m)$, $r > \frac{n}{2}$, and by $\sigma(D^2)$ the spectrum.

Lemma 4.4.

a. $W(t, m, p)$ is for $t > 0$ smooth in all variables.

b. For any $T > 0$ and sufficiently small $\varepsilon > 0$ there exists $C > 0$ such that

$$|W(t, m, p)| \leq e^{-(t-\varepsilon) \inf \sigma(D^2)} \cdot C \cdot C(m) \cdot C(p) \text{ for all } t \in]T, \infty[. \quad (4.3)$$

c. Similar estimates hold for $(D_m^i D_p^j W)(t, m, p)$.

Proof.

a. First one shows W is continuous, which follows from $\langle \delta(m), \cdot \rangle$ continuous in m and $e^{-tD^2} \delta(p)$ continuous in t and p . Then one applies elliptic regularity.

b. Write

$$|\langle \delta(m), e^{-tD^2} \delta(p) \rangle| = |\langle (1 + D^2)^{-\frac{r}{2}} \delta(m), (1 + D^2)^r e^{-tD^2} (1 + D^2)^{\frac{r}{2}} \delta(p) \rangle|.$$

c. Follows similar as b. □

Lemma 4.5. *For any $\varepsilon > 0, T > 0, \delta > 0$ there exists $C > 0$ such that for $r > 0, m \in M, T > t > 0$ holds*

$$\int_{M \setminus B_r(m)} |W(t, m, p)|^2 dp \leq C \cdot C(m) \cdot e^{-\frac{(r-\varepsilon)^2}{(4+\delta)t}}. \quad (4.4)$$

A similar estimate holds for $D_m^i D_p^j W(t, m, p)$.

We refer to [2] for the proof.

Lemma 4.6. *For any $\varepsilon > 0, T > 0, \delta > 0$ there exists $C > 0$ such that for all $m, p \in M$ with $\text{dist}(m, p) > 2\varepsilon, T > t > 0$ holds*

$$|W(t, m, p)|^2 \leq C \cdot C(m) \cdot C(p) \cdot e^{-\frac{(\text{dist}(m, p) - \varepsilon)^2}{(4+\delta)t}}. \quad (4.5)$$

A similar estimate holds for $D_m^i D_p^j W(t, m, p)$.

We refer to [2] for the proof.

Proposition 4.7. *Assume (M^n, g) with (I) and (B_k) , (E, ∇) with (B_k) , $k \geq r > \frac{n}{2} + 1$. Then all estimates in 4.4, 4.5, 4.6 hold with constants.*

Proof. From the assumptions $H^r(E) \cong W^r(E)$ and $\sup_m C(m) = C = \text{global Sobolev constant for } W^r(E)$. \square

Let $U \subset M$ be precompact, open, (M^+, g^+) closed with $U \subset M^+$ isometrically and $E^+ \rightarrow M^+$ a Clifford bundle with $E^+|_U \cong E|_U$ isometrically. Denote by $W^+(t, m, p)$ the heat kernel of e^{-tD^+} .

Lemma 4.8. *Assume $\varepsilon > 0, T > 0, \delta > 0$. Then there exists $C > 0$ such that for all $T > t > 0, m, p \in U$ with $B_{2\varepsilon}(m), B_{2\varepsilon}(p) \subset U$ holds*

$$|W(t, m, p) - W^+(t, m, p)| \leq C \cdot e^{-\frac{\varepsilon^2}{(4+\delta)t}} \quad (4.6)$$

We refer to [2] for the simple proof. \square

Corollary 4.9. *$\text{tr}W(t, m, m)$ has for $t \rightarrow 0^+$ the same asymptotic expansion as $\text{tr}W^+(t, m, m)$.* \square

5 Trace class property under variation of the Clifford connection

We come now to the first main result of this paper.

Theorem 5.1. *Assume $(E, \nabla) \rightarrow (M^n, g)$, (M^n, g) with (I) and (B_k) (E, ∇) with (B_k) , $k \geq r > n + 2, n \geq 2$, $\nabla' \in \text{comp}(\nabla) \cap C_E(B_k) \subset C_E^{1,r}(B_k)$, $D = D(g, \nabla), D' = D'(g, \nabla')$, generalized Dirac operators. Then*

$$e^{-tD^2} - e^{-tD'^2}$$

is for $t > 0$ trace class operator and its trace norm is uniformly bounded on compact t -intervals $[a_0, a_1], a_0 > 0$.

Remark. The condition $\nabla' \in \text{comp}(\nabla) \cap C_E(B_k) \subset C_E^{1,r}(B_k)$, i. e. $\nabla' \in \text{comp}(\nabla)$ and additionally ∇' smooth and satisfying (B_k) can be weakened to $\nabla' \in \text{comp}(\nabla) \subset C_E^{1,r}(B_k)$. The main reason for this is that we can write $\nabla' = \nabla'_0 + (\nabla' - \nabla'_0)$, $\nabla'_0 \in C_E(B_k)$, $|\nabla' - \nabla'_0|_{1,r,\nabla} < \varepsilon$. Then one can reestablish the whole Sobolev theory etc. extensively using the module structure theorem. We refer to the forthcoming paper [8]. \square

The proof of theorem 5.1 will occupy the remaining part of this section. We always assume the assumptions of 5.1. According to (3.11),

$$\mathcal{D}_D = \mathcal{D}_{D'}, \quad \mathcal{D}_{D^2} = \mathcal{D}_{D'^2}.$$

Lemma 5.2. *Assume $t > 0$. Then*

$$e^{-tD^2} - e^{-tD'^2} = \int_0^t e^{-sD^2} (D'^2 - D^2) e^{-(t-s)D'^2} ds. \quad (5.1)$$

Proof. (5.1) means at heat kernel level

$$W(t, m, p) - W'(t, m, p) = - \int_0^t \int_M (W(s, m, q), (D^2 - D'^2) W'(t-s, q, p))_q dq ds, \quad (5.2)$$

where $(\cdot, \cdot)_q$ means the fibrewise scalar product at q and $dq = d\text{vol}_q(g)$. Hence for (5.1) we have to prove (5.2). (5.2) is an immediate consequence of Duhamel's principle. Only for completeness, we present the proof of (5.2), which is the last of the following 7 facts and implications.

1. For $t > 0$ is $W(t, m, p) \in L_2(M, E, dp) \cap \mathcal{D}_D^2$.
2. If $\Phi, \Psi \in \mathcal{D}_D^2$ then $\int (D^2 \Phi, \Psi) - (\Phi, D^2 \Psi) d\text{vol} = 0$ (Greens formula).
3. $((D^2 + \frac{\partial}{\partial \tau}) \Phi(\tau, g) \Psi(t - \tau, q))_q - (\Phi(\tau, g), (D^2 + \frac{\partial}{\partial t}) \Psi(t - \tau, q))_q = (D^2 \Phi(\tau, q), \Psi(t - \tau, q))_q - (\Phi(\tau, q), D^2 \Psi(t - \tau, q))_q + \frac{\partial}{\partial \tau} (\Phi(\tau, g), \Psi(t - \tau, q))_q$.
4. $\int_{\alpha}^{\beta} \int_M ((D^2 + \frac{\partial}{\partial \tau}) \Phi(\tau, q), \Psi(t - \tau, q))_q - (\Phi(\tau, q), (D^2 + \frac{\partial}{\partial t}) \Psi(t - \tau, q))_q dq d\tau = \int_M [(\Phi(\beta, q), \Psi(t - \beta, q))_q - (\Phi(\alpha, q), \Psi(t - \alpha, q))_q] dq$.
5. $\Phi(t, q) = W(t, m, q), \Psi(t, q) = W'(t, q, p)$ yields $-\int_{\alpha}^{\beta} \int_M (W(\tau, m, q), (D^2 + \frac{\partial}{\partial t}) W'(t - \tau, q, p))_q dq d\tau = \int_M [(W(\beta, m, q), W'(t - \beta, q, p))_q - (W(\alpha, m, q), W'(t - \alpha, q, p))_q] dq$.
6. Performing $\alpha \rightarrow 0^+, \beta \rightarrow t^-$ in 5. yields $-\int_0^t \int_M (W(s, m, q), (D^2 + \frac{\partial}{\partial t}) W'(t - s, q, p))_q dq ds = W(t, m, p) - W'(t, m, p)$.
7. Finally, using $D^2 + \frac{\partial}{\partial t} = D^2 - D'^2 + D'^2 + \frac{\partial}{\partial t}$ and $(D'^2 + \frac{\partial}{\partial t}) W' = 0$ we obtain $W(t, m, p) - W'(t, m, p) = - \int_0^t \int_M (W(s, m, q), (D^2 - D'^2) W'(t - s, q, p))_q dq ds$ which is (5.2). □

If we write $D'^2 - D^2 = D'(D' - D) + (D' - D)D$ then

$$\begin{aligned}
e^{-tD^2} - e^{-tD'^2} &= \int_0^t e^{-sD^2} D'(D' - D) e^{-(t-s)D'^2} ds + \\
&+ \int_0^t e^{-sD^2} (D' - D) D e^{-(t-s)D'^2} ds = \\
&= \int_0^t e^{-sD^2} D' \eta e^{-(t-s)D'^2} ds + \\
&+ \int_0^t e^{-sD^2} \eta D e^{-(t-s)D'^2} ds,
\end{aligned}$$

where $\eta = \eta^{op}$ in the sense of section 3, $\eta^{op}(\Psi)|_x = \sum_{i=1}^n e_i \cdot \eta_{e_i}(\Psi)$ and $|\eta^{op}|_x \leq C \cdot |\eta|_x$, C independent of x . We split $\int_0^t = \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t$,

$$e^{-tD^2} - e^{-tD'^2} = \int_0^{\frac{t}{2}} e^{-sD^2} D' \eta e^{-(t-s)D'^2} ds + \quad (I_1)$$

$$+ \int_0^{\frac{t}{2}} e^{-sD^2} \eta D e^{-(t-s)D'^2} ds + \quad (I_2)$$

$$+ \int_{\frac{t}{2}}^t e^{-sD^2} D' \eta e^{-(t-s)D'^2} ds + \quad (I_3)$$

$$+ \int_{\frac{t}{2}}^t e^{-sD^2} \eta D e^{-(t-s)D'^2} ds. \quad (I_4)$$

We want to show that each integral $(I_1) - (I_4)$ is a product of Hilbert-Schmidt operators and to estimate its Hilbert-Schmidt norm. Consider the integrands of (I_3) resp. (I_4) . Applying a Leibniz type rule in

$$e^{-sD^2} D' \eta e^{-(t-s)D'^2},$$

we have to estimate

$$(e^{-sD^2} \nabla' \eta) \circ (e^{-(t-s)D'^2}) \quad (5.3)$$

and

$$(e^{-sD^2} \eta) \circ (D' e^{-(t-s)D'^2}) \quad (5.4)$$

Similarly for (I_4)

$$\begin{aligned} e^{-sD^2} \eta D e^{-(t-s)D'^2} &= e^{-sD^2} \eta ((D - D') + D') e^{-(t-s)D'^2} = \\ &= \left(-e^{-sD^2} \eta^2 \right) \circ \left(e^{-(t-s)D'^2} \right) + \end{aligned} \quad (5.5)$$

$$+ \left(e^{-sD^2} \eta \right) \circ \left(D' e^{-(t-s)D'^2} \right). \quad (5.6)$$

Then according to 4.2,

$$|e^{-(t-s)D'^2}|_{L_2 \rightarrow H^1} \leq C \cdot (t-s)^{-\frac{1}{2}} \quad (5.7)$$

and

$$|D' e^{-(t-s)D'^2}|_{L_2 \rightarrow L_2} \leq |D'|_{H^1 \rightarrow L_2} |e^{-(t-s)D'^2}|_{L_2 \rightarrow H^1} \leq C' \cdot (t-s)^{\frac{1}{2}}. \quad (5.8)$$

(5.7) and (5.8) estimate the right hand factors in (5.3) – (5.6). Start now with the left hand factor in (5.6), $e^{-sD^2} \eta$ and write

$$e^{-sD^2} \eta = e^{-\frac{s}{2}D^2} \circ e^{-\frac{s}{2}D^2} \eta = (e^{-\frac{s}{2}D^2} \circ f) \circ (f^{-1} \circ e^{-\frac{s}{2}D^2} \eta). \quad (5.9)$$

Here f shall be a scalar function which acts by multiplication. The main point is the right choice of f . $e^{-\frac{s}{2}D^2} f$ has the integral kernel

$$W\left(\frac{s}{2}, m, p\right) f(p) \quad (5.10)$$

and $f^{-1} e^{-\frac{s}{2}D^2}$ has the kernel

$$f^{-1}(m) W\left(\frac{s}{2}, m, p\right) \eta(p) \quad (5.11)$$

We have to make a choice such that (5.10), (5.11) are square integrable over $M \times M$ and that their L_2 -norm is uniformly bounded.

We decompose the L_2 -norm of (5.10) as

$$\begin{aligned} \int_M \int_M |W\left(\frac{s}{2}, m, p\right)|^2 |f(m)|^2 dm dp &= \\ \int_M \int_{\text{dist}(m,p) \geq c} |W\left(\frac{s}{2}, m, p\right)|^2 |f(m)|^2 dp dm &+ \end{aligned} \quad (5.12)$$

$$\int_M \int_{\text{dist}(m,p) < c} |W\left(\frac{s}{2}, m, p\right)|^2 |f(m)|^2 dp dm \quad (5.13)$$

We obtain from 4.4 for $s \in]\frac{t}{2}, t[$

$$(5.13) \leq \int_M C_1 |f(m)|^2 \text{vol} B_c(m) dm \leq C_2 \int_M |f(m)|^2 dm$$

and from 4.5

$$\int_M \int_{\text{dist}(m,p) \geq c} |W\left(\frac{s}{2}, m, p\right)|^2 |f(m)|^2 dp dm \leq \int_M C_1 e^{-\frac{(r-s)^2}{4+\delta} \frac{2}{s}} |f(m)|^2 dm \leq$$

$$\leq C_1 \cdot e^{-\frac{(c-\varepsilon)^2}{4+\delta} \frac{2}{s}} \int_M |f(m)|^2 dm, \quad c > \varepsilon. \quad (5.14)$$

Hence the estimate of $\int_M \int_M |W(\frac{s}{2}, m, p)|^2 |f(m)|^2 dp dm$ for $s \in [\frac{t}{2}, t]$ is done if

$$\int_M |f(m)|^2 dm < \infty.$$

For (5.11) we have to estimate

$$\int_M \int_M |f(m)|^{-2} |(W(\frac{s}{2}, m, p), \eta^{op}(p) \cdot)_p|^2 dp dm \quad (5.15)$$

We recall a simple fact in Hilbert spaces. Let X be a Hilbert space, $x \in X, x \neq 0$. Then $|x| = \sup_{|y|=1} |\langle x, y \rangle|$,

$$|x|^2 = \left(\sup_{|y|=1} |\langle x, y \rangle| \right)^2. \quad (5.16)$$

This follows from $|\langle x, y \rangle| \leq |x| \cdot |y|$ and equality for $y = \frac{x}{|x|}$. We apply this to $E \rightarrow M$, $X = L_2(M, E, dp)$, $x = x(m)(p) = (W(t, m, p), \eta^{op}(p) \cdot)_p = \eta^{op}(p) \circ W(t, m, p)$ and have to estimate

$$\sup_{\substack{\Phi \in C_c^\infty(E) \\ |\Phi|_{L_2} = 1}} N(\Phi) = \sup_{\substack{\Phi \in C_c^\infty(E) \\ |\Phi|_{L_2} = 1}} |\langle \delta(m), e^{-tD^2} \eta^{op} \Phi \rangle| \quad (5.17)$$

According to 4.5, for $t > 0$

$$W(t, m, \cdot) \in H^{\frac{r}{2}}(E), \quad |W(t, m, \cdot)|_{H^{\frac{r}{2}}} \leq C_1(t). \quad (5.18)$$

Hence we can restrict in (5.17) to

$$\sup_{\substack{\Phi \in C_c^\infty(E) \\ |\Phi|_{L_2} = 1 \\ |\Phi|_{H^{\frac{r}{2}}} \leq C_1}} N(\Phi) \quad (5.19)$$

In the sequel we estimate (5.19). For doing this, we recall some simple facts concerning the wave equation

$$\frac{\partial \Phi_s}{\partial s} = iD\Phi_s, \quad \Phi_0 = \Phi, \quad \Phi \in C^1 \text{ with compact support.} \quad (5.20)$$

It is well known that (5.20) has a unique solution Φ_s which is given by

$$\Phi_s = e^{isD} \Phi \quad (5.21)$$

and

$$\text{supp } \Phi_s \subset U_{|s|} \text{ (supp } \Phi) \quad (5.22)$$

$U_{|s|} = |s|$ - neighborhood. Moreover,

$$|\Phi_s|_{L_2} = |\Phi|_{L_2}, \quad |\Phi_s|_{H^{\frac{r}{2}}} = |\Phi|_{H^{\frac{r}{2}}} \quad (5.23)$$

We fix a uniformly locally finite cover $\mathcal{U} = \{U_\nu\}_\nu = \{B_d(x_\nu)\}_\nu$ by normal charts of radius $d < r_{inj}(M, g)$ and associated decomposition of unity $\{\varphi_\nu\}_\nu$ satisfying

$$|\nabla^i \varphi_\nu| \leq C \text{ for all } \nu, \quad 0 \leq i \leq k+2 \quad (5.24)$$

Write

$$\begin{aligned} N(\Phi) &= |\langle \delta(m), e^{r-tD^2} \eta^{op} \Phi \rangle| \\ &= \frac{1}{\sqrt{4\pi t}} \left| \langle \delta(m), \int_{-\infty}^{+\infty} e^{\frac{-s^2}{4t}} e^{isD} (\eta^{op} \Phi) ds \rangle \right| = \\ &= \frac{1}{\sqrt{4\pi t}} \left| \int_{-\infty}^{+\infty} e^{\frac{-s^2}{4t}} (e^{isD} \eta^{op} \Phi)(m) ds \right|. \end{aligned} \quad (5.26)$$

We decompose

$$\eta^{op} = \sum_\nu \varphi_\nu \eta^{op} \Phi. \quad (5.27)$$

(5.27) is a locally finite sum, (5.20) linear. Hence

$$(\eta^{op})_s = \sum_\nu (\varphi_\nu \eta^{op} \Phi)_s. \quad (5.28)$$

Denote as above

$$| \cdot |_{p,i} \equiv | \cdot |_{W^{p,i}},$$

in particular

$$| \cdot |_{2,i} \equiv | \cdot |_{W^{2,i}} \sim | \cdot |_{H^i}, \quad i \leq k. \quad (5.29)$$

Then we obtain from (5.23), (5.28), (2.1)

$$\begin{aligned} |(\varphi_\nu \eta^{op} \Phi)_s|_{H^{\frac{r}{2}}} &= |\varphi_\nu \eta^{op} \Phi|_{H^{\frac{r}{2}}} \leq C_2 |\varphi_\nu \eta^{op} \Phi|_{2, \frac{r}{2}} \leq \\ &\leq C_3 |\eta^{op} \Phi|_{2, \frac{r}{2}, U_\nu} \leq C_4 |\eta|_{2, \frac{r}{2}, U_\nu} \leq C_5 |\eta|_{1, r-1, U_\nu} \end{aligned} \quad (5.30)$$

since $r-1-\frac{n}{1} \geq \frac{r}{2}-\frac{n}{2}$, $r-1 \geq \frac{r}{2}$, $2 \geq 1$ for $r > n+2$ and $|\Phi|_{H^{\frac{r}{2}}} \leq C_1$. This yields together with (2.3), (2.13) the estimate

$$\begin{aligned} |(\eta^{op} \Phi)_s(m)| &\leq C_6 \cdot \sum_{\substack{\nu \\ m \in U_{|s|}(U_\nu)}} |(\varphi_\nu \eta^{op} \Phi)_s|_{2, \frac{r}{2}} \leq \end{aligned}$$

$$\begin{aligned}
&\leq C_7 \cdot \sum_{\substack{\nu \\ m \in U_{|s|}(U_\nu)}} |\eta|_{1,r-1,U_\nu} \leq C_8 \cdot |\eta|_{1,r-1,B_{2d+|s|}(m)} = \\
&= C_8 \cdot \text{vol}(B_{2d+|s|}(m)) \cdot \left(\frac{1}{\text{vol} B_{2d+|s|}(m)} \cdot |\eta|_{1,r-1,B_{2d+|s|}(m)} \right).
\end{aligned} \tag{5.31}$$

There exist constants A and B , independent of m s. t.

$$\text{vol}(B_{2d+|s|}(m)) \leq A \cdot e^{B|s|}.$$

Write

$$e^{-\frac{s^2}{4t}} \cdot \text{vol}(B_{2d+|s|}(m)) \leq C_9 \cdot e^{-\frac{9}{10} \frac{s^2}{4t}}, \tag{5.32}$$

thus obtaining

$$N(\Phi) \leq C_{10} \int_0^\infty e^{-\frac{9}{10} \frac{s^2}{4t}} \left(\frac{1}{\text{vol} B_{2d+|s|}(m)} \cdot |\eta|_{1,r-1,B_{2d+|s|}(m)} \right) ds.$$

Now we apply (2.7) with $R = 3d + s$ and infer

$$\begin{aligned}
&\int_M \frac{1}{\text{vol} B_{2d+|s|}(m)} \cdot |\eta|_{1,r-1,B_{2d+|s|}(m)} dm \leq \\
&\leq |\eta|_{1,r-1} + C(3d + s) \cdot (2d + s) |\nabla \eta|_{1,r-1} \leq \\
&\leq |\eta|_{1,r-1} + C(3d + s) \cdot (2d + s) |\eta|_{1,r}.
\end{aligned} \tag{5.33}$$

$C(3d + s)$ depends on $3d + s$ at most linearly exponentially, i. e.

$$C(3d + s) \leq A_1 e^{B_1(3d+s)}.$$

This implies

$$\begin{aligned}
&\int_0^\infty e^{-\frac{9}{10} \frac{s^2}{4t}} \int_M \frac{1}{\text{vol} B_{2d+|s|}(m)} \cdot |\eta|_{1,r-1,B_{2d+|s|}(m)} dm ds \leq \\
&\leq \int_0^\infty e^{-\frac{9}{10} \frac{s^2}{4t}} (|\eta|_{1,r-1} + C(3d + s) \cdot (2d + s) |\eta|_{1,r}) ds < \infty.
\end{aligned} \tag{5.34}$$

The function

$$\begin{aligned}
&\mathbf{R}_+ \times M \rightarrow \mathbf{R}, \\
&(s, m) \rightarrow e^{-\frac{9}{10} \frac{s^2}{4t}} \left(\frac{1}{\text{vol} B_{2d+|s|}(m)} \cdot |\eta|_{1,r-1,B_{2d+|s|}(m)} \right)
\end{aligned}$$

is measurable, nonnegative, the integrals (5.33), (5.34) exist, hence according to the principle of Tonelli, this function is 1-summable, the Fubini theorem is applicable and

$$\tilde{\eta}(m) := C_{10} \cdot \int_0^\infty e^{-\frac{9}{10} \frac{s^2}{4t}} \left(\frac{1}{\text{vol} B_{2d+|s|}(m)} \cdot |\eta|_{1,r-1, B_{2d+|s|}(m)} \right) ds$$

is (for $\eta \neq 0$) everywhere $\neq 0$ and 1-summable. We proved

$$\int |(W(t, m, p), \eta^{op} \cdot)_p|^2 dp \leq \tilde{\eta}(m)^2. \quad (5.35)$$

Now we set

$$f(m) = (\tilde{\eta}(m))^{\frac{1}{2}} \quad (5.36)$$

and infer $f(m) \neq 0$ everywhere, $f \in L_2$ and

$$\begin{aligned} \int_M \int_M f(m)^{-2} |(W(\frac{t}{2}, m, p), \eta^{op} \cdot)_p|^2 dp dm &\leq \\ &\leq \int_M \frac{1}{\tilde{\eta}(m)} \tilde{\eta}(m)^2 dm = \int_M \tilde{\eta}(m) dm \leq C_{11}(t) \cdot \sqrt{t}, \end{aligned} \quad (5.37)$$

since $\int_{-\infty}^{+\infty} e^{-ax^2} dx = \frac{\sqrt{\pi}}{\sqrt{a}}$, $\int_0^\infty e^{-ax^2} dx = \frac{\sqrt{\pi}}{2\sqrt{a}}$, in particular for $t \in [a_0, a_1]$, $a_0 > 0$,

$$\sup_{t \in [a_0, a_1]} C_{11}(t) \cdot \sqrt{t} = C_{12}(a_0, a_1) < \infty. \quad (5.38)$$

As shown above, the integrals (5.12), (5.13) can be estimated by constants $C_{13}(a_0, a_1)$, $C_{14}(a_0, a_1)$. Finally we use for A of trace class, B bounded.

$$|A \cdot B|_1 \leq |A|_1 \cdot |B|_{op} \quad (5.39)$$

and obtain

$$\begin{aligned} &\left| \int_{\frac{t}{2}}^t (e^{-sD^2} \eta D' e^{-(t-s)D'^2}) ds \right|_1 \leq \\ &\leq \sup_{s \in [\frac{t}{2}, t]} |e^{-sD^2} \eta|_1 \int_{\frac{t}{2}}^t |D' e^{-(t-s)D'^2}|_{L_2 \rightarrow L_2} ds \leq \\ &\leq C_{12}(\frac{t}{2}, t) (C_{13}(\frac{t}{2}, t) + C_{14}(\frac{t}{2}, t)) \cdot C \cdot \sqrt{t} = C_{15}(\frac{t}{2}, t), \end{aligned} \quad (5.40)$$

i. e. the operator (5.6) is of trace class and for $t \in [a_0, a_1]$, $a_0 > 0$, its trace norm is uniformly bounded.

We proceed with the expression (5.5). The only distinction here is the appearance of $(\eta^{op})^2$ instead of η^{op} . Then we estimate as in (5.6), replacing η^{op} by $(\eta^{op})^2$. The estimates become even better. Or we write

$$e^{-sD^2}(\eta^{op})^2 e^{-(t-s)D'^2} = \left(e^{-sD^2} \eta^{op}\right) \circ \left(\eta^{op} e^{-(t-s)D'^2}\right) \quad (5.41)$$

Here η^{op} acts in $\eta^{op} e^{-(t-s)D'^2}$ as a bounded operator according to the module structure theorem. Hence (I_4) is done. (I_3) can be settled in exactly the same manner. In (I_2) and (I_1) we change the role of the factors. Decompose the integrand of (I_2) as

$$\begin{aligned} e^{-sD^2} \eta D e^{-(t-s)D'^2} &= \\ &= -e^{-sD^2} \circ \eta \circ (\eta e^{-(t-s)\frac{D'^2}{2}} \cdot f^{-1}) \circ (f \cdot e^{-(t-s)\frac{D'^2}{2}}) + \end{aligned} \quad (5.42)$$

$$+ e^{-sD^2} \circ (\eta \cdot e^{-(t-s)\frac{D'^2}{2}} \cdot f^{-1}) \circ (f \cdot D' e^{-(t-s)\frac{D'^2}{2}}). \quad (5.43)$$

According to (5.7), (5.8) and the module structure or embedding theorem,

$$e^{-sD^2}, \eta^{op}, e^{-sD^2} \circ \eta^{op} \text{ are bounded for } s \leq \frac{t}{2}. \quad (5.44)$$

The terms

$$\eta^{op} \circ e^{-(t-s)\frac{D'^2}{2}} \cdot f^{-1}, \quad (5.45)$$

$$f \cdot e^{-(t-s)\frac{D'^2}{2}} \quad (5.46)$$

and

$$f \cdot D' e^{-(t-s)\frac{D'^2}{2}} \quad (5.47)$$

in $s \in [0, \frac{t}{2}]$ can be estimated as

$$f^{-1} \cdot e^{-\frac{s}{2}D^2} \eta^{op}$$

and

$$e^{-\frac{s}{2}D^2} f$$

for $s \in [\frac{t}{2}, t]$.

Finally the integrand of (I_1) can be written

$$\begin{aligned} e^{-sD^2} D' \eta e^{-(t-s)D'^2} &= \\ &= e^{-sD^2} \circ ((\nabla'^{op} \eta^{op}) e^{-(t-s)\frac{D'^2}{2}} \cdot f^{-1}) \circ (f \cdot e^{-(t-s)\frac{D'^2}{2}}) + \end{aligned} \quad (5.48)$$

$$+ e^{-sD^2} \circ (\eta^{op} \cdot e^{-(t-s)\frac{D'^2}{2}} \cdot f^{-1}) \circ (f \cdot D' e^{-(t-s)\frac{D'^2}{2}}). \quad (5.49)$$

The two right terms of (5.48), (5.49) can be estimated as (5.45) – (5.47).

This finishes the proof of theorem 5.1. \square

For our applications in section 7 we need still the trace class property of

$$D e^{-tD^2} - D' e^{-tD'^2} = e^{-tD^2} D - e^{-tD'^2} D'.$$

Consider

$$\begin{aligned} e^{-tD^2}D - e^{-tD'^2}D' &= e^{-tD^2}(D - D') + (e^{-tD^2} - e^{-tD'^2})D' = \\ &= e^{-tD^2}(D - D') + \int_0^t e^{-sD^2}D'(D' - D)D'e^{-(t-s)D'^2}ds + \int_0^t e^{-sD^2}(D' - D)DD'e^{-(t-s)D'^2}ds. \end{aligned}$$

Now

$$e^{-tD^2}(D - D') = -e^{-tD^2}\eta^{op} = -\left(e^{-t\frac{D^2}{2}}f\right) \circ \left(f^{-1}e^{-t\frac{D^2}{2}}\eta^{op}\right),$$

f as in (5.36), and we are done,

$$|e^{-tD^2}(D - D')|_1 \leq C \cdot \sqrt{t}. \quad (5.50)$$

Decompose

$$\int_0^t e^{-sD^2}D'(D' - D)D'e^{-(t-s)D'^2}ds = \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t.$$

The estimate of $\int_{\frac{t}{2}}^t$ amounts to that of

$$\left(e^{-\frac{s}{2}\frac{D^2}{2}} \cdot f\right) \circ \left(f^{-1}e^{-\frac{s}{2}D^2} \circ \nabla'\eta\right), \quad (5.51)$$

$$D'e^{-(t-s)D'^2}, \quad (5.52)$$

$$\left(e^{-\frac{s}{2}\frac{D^2}{2}}f\right) \circ \left(f^{-1}e^{-s\frac{D^2}{2}}\nabla'\eta\right), \quad (5.53)$$

$$D'^2e^{-(t-s)D'^2} \quad (5.54)$$

(5.51) can be estimated as (5.9), assuming $|\eta|_{1,r+1} < \infty$, $k \geq r > n + 2$, (5.52) as (5.7), (5.53) as (5.9). A small difficulty arises with (5.54) since

$$|D'^2e^{-(t-s)D'^2}|_{L_2 \rightarrow H_1} \leq C((t-s)^{-\frac{1}{2}})^2. \quad (5.55)$$

But, considering (5.37), we see that $|(5.53)|_1$ generates a factor $(t-s)^{\frac{1}{2}}$ and we obtain $(t-s)^{-\frac{1}{2}}$ for integration which doesn't cause any trouble. Quite similar we handle

$$\begin{aligned} &\int_0^t e^{-sD^2}(D - D')DD'e^{-(t-s)D'^2}ds = \\ &= -\int_0^t e^{-sD^2}(D - D')^2D'e^{-(t-s)D'^2}ds + e^{-sD^2}(D - D')D'^2e^{-(t-s)D'^2}ds \\ &= -\int_0^t (e^{-sD^2}\eta^2)D'e^{-(t-s)D'^2}ds + \int_0^t e^{-sD^2}\eta D'e^{-(t-s)D'^2}ds. \end{aligned}$$

We decompose $\int_0^t = \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t$ and proceed as in (5.51) – (5.55). Hence we proved

Theorem 5.3. *Assume $(E, \nabla) \rightarrow (M^n, g)$ with $(I), (B_k), (E, \nabla)$ with (B_k) $k \geq r > n + 3$, $n \geq 2$, $\nabla' \in \text{comp}(\nabla) \cap \mathcal{C}_E(B_k) \subset \mathcal{C}_E^{1,r}(B_k)$, $D = D(g, \nabla)$, $D' = D(g, \nabla')$ generalized Dirac operators. Then*

$$e^{-tD^2} - e^{-tD'^2}$$

and

$$De^{-tD^2} - D'e^{-tD'^2}$$

are for $t > 0$ trace class operators and their trace norm is uniformly bounded on compact t -intervalls $[a_0, a_1], a_0 > 0$. \square

6 Trace class property for additional variation of the metric

As we know from the definition, $D = D(g, \nabla) = D(g, E, \nabla)$. In section 5 we considered $D' = D(g, E, \nabla')$. More general, we should consider $D' = D(g', \nabla')$. But at the first glance, this does not make sense. Change of g changes the Clifford algebra Cl_m , we have now $Cl(T_m, g'_m)$ and hence have to consider modules of $Cl(T_m, g'_m)$. A Clifford bundle associated to g' must consist fibrewise of such modules, we arrive at a new bundle E' . E' can have a new fibre metric h' . Nevertheless, locally E and E' are isomorphic. Motivated by the consideration that the metric parameters $g \rightarrow g'$, $h \rightarrow h'$ move smoothly, we assume that $E \rightarrow E'$ moves smoothly, $E \cong E'$ as smooth vector bundle. Hence we identify E and E' , keeping in mind that the fibres E_m have different module structures over different algebras. Such a module structure is given by a section \cdot of $\Gamma(\text{Hom}(TM \times E, E))$. Endowing TM with g, ∇^g , E with $h, \nabla, \text{Hom}(TM \times E, E)$ becomes a Riemannian vector bundle. Hence

$$W^{p,r}(\text{Hom}(TM \times E, E), g, h, \nabla)$$

is well defined. Assuming $g, g', h, h', \nabla, \nabla'$ such that

$$W^{1,r}(\text{Hom}(TM \times E, E), g, h, \nabla) \cong W^{1,r}(\text{Hom}(TM \times E, E), g', h', \nabla')$$

then the condition

$$\cdot -' \in W^{1,r}(\text{Hom}(TM \otimes E, E), g', h', \nabla') \quad (\text{Clm})$$

makes sense.

We make in this section the following

Assumptions. $(E, h, \nabla) \rightarrow (M^n, g), (E, h', \nabla') \rightarrow (M^n, g')$ Clifford bundles with $(I), (B_k(M)), (B_k(E)), k \geq r > n + 2$

$$g' \in \text{comp}(g) \cap \mathcal{M}(I, B_k) \subset \mathcal{M}^{1,r}(I, B_k). \quad (6.1)$$

$$h \text{ and } h' \text{ quasi isometric and } |h - h'|_{g,h,\nabla,1,r} < \infty, |h - h'|_{g',h',\nabla',1,r} < \infty. \quad (6.2)$$

$$|\nabla - \nabla'|_{g,h,\nabla,1,r} < \infty. \quad (6.3)$$

$$|\nabla - \nabla'|_{g',h',\nabla',1,r} < \infty. \quad (6.4)$$

and

$$\cdot -' \in W^{1,r}(\text{Hom}(TM \otimes E, E), g', h', \nabla') \quad (\text{Clm})$$

Here we understand $\nabla - \nabla'$ as a 1-form with values in $\text{End } E$. (6.3) means

$$\nabla - \nabla' \in \Omega^{1,1,r}(\text{End } E, g, h, \nabla^g, \nabla) \quad (6.5)$$

and

$$|\nabla - \nabla'|_{g,h,\nabla,1,r} = \int_M \sum_{i=0}^r |\nabla^i(\nabla - \nabla')|_{g,x} d\text{vol}_x(g). \quad (6.6)$$

The main result of this section shall be formulated as follows.

Theorem 6.1. *Let $(E, h, \nabla) \rightarrow (M^n, g), (E, h', \nabla') \rightarrow (M^n, g')$ be Clifford bundles with $(I), (B_k(M)), (B_k(E)), k \geq r > n + 2$, and (6.1)–(6.4), (Clm). Let $D = D(g, h, \nabla), D' = D(g', h', \nabla')$. Then*

$$e^{-tD^2} - e^{-tD'^2}$$

is of trace class and the trace norm is uniformly bounded on compact t -intervalls $[a_0, a_1], a_0 > 0$.

Remarks.

1. We shall see below that e^{-tD^2} and $e^{-tD'^2}$ act between the same spaces.
2. For $g = g', h = h'$ we obtain back theorem 5.1. □

The proof of 6.1. occupies the remaining point of this section. We always assume the hypotheses of 6.1.

Lemma 6.2. $W^{1,i}(E, g, h, \nabla) = W^{1,i}(E, g', h', \nabla'), 0 \leq i \leq r$ as equivalent Banach spaces.

Corollary 6.3. $W^{2,j}(E, g, h, \nabla) = W^{2,j}(E, g', h', \nabla'), 0 \leq j \leq \frac{r}{2}$ as equivalent Hilbert spaces. In particular,

$$L_2((M, E), g, h) = L_2((M, E), g', h'). \quad (6.7)$$

Corollary 6.4. $H^j(E, D) \cong H^j(E, D'), 0 \leq j \leq \frac{r}{2}$.

Proof of (6.2). This is well known for $h = h', \nabla' \in \text{comp}(\nabla)$. But concerning ∇, ∇' and h, h' the only two facts needed in the proof are just (6.3) (which is reformulated as (6.5)), (6.4) and

the equivalence of pointwise norms. The latter follows from (6.1), (6.2). Into higher derivatives enter $(\nabla^g)^i, (\nabla^{g'})^j, i, j \leq r-1$. The conditions

$$|\nabla^g - \nabla^{g'}|_{g,1,r-1} < \infty \quad |\nabla^g - \nabla^{g'}|_{g',1,r-1} < \infty$$

follow from $g' \in \text{comp}(g)$. □

6.2. has a parallel version for the endomorphism bundle $\text{End } E$.

Lemma 6.5. $\Omega^{1,1,i}(\text{End } E, g, h, \nabla) \cong \Omega^{1,1,i}(\text{End } E, g', h', \nabla'), \quad 0 \leq i \leq r.$ □

Corollary 6.6. $\Omega^{1,2,j}(\text{End } E, g, h, \nabla) \cong \Omega^{1,2,j}(\text{End } E, g', h', \nabla'), \quad 0 \leq j \leq \frac{r}{2}.$ □

We obtain

$$e^{-tD^2}, e^{-tD'^2} : L_2((M, E), g, h) \rightarrow H^j(E, D), \quad 0 \leq j \leq \frac{r}{2} \quad (6.8)$$

Hence

$$e^{-tD^2} - e^{-tD'^2}$$

is well defined. Our next task is to obtain an explicit expression for $e^{-tD^2} - e^{-tD'^2}$. For this we must modify Duhamel's principle slightly. The steps 1. – 5. in the proof of 5.2. remain unchanged. We perform them for $(\cdot, \cdot)_q = h_q(\cdot, \cdot)$, $dq = \text{vol}_q(g) \equiv dq(g)$. Then 5. reads as

$$\begin{aligned} & - \int_{\alpha}^{\beta} \int_M h_q(W(\tau, m, q), (D^2 + \frac{\partial}{\partial t})W'(t - \tau, q, p)) dq(g) d\tau = \\ & = \int_M \left[h_q(W(\beta, m, q), W'(t - \beta, q, p)) - \right. \\ & \quad \left. - h_q(W(\alpha, m, q), W'(t - \alpha, q, p)) \right] dq(g). \end{aligned} \quad (6.9)$$

Performing $\alpha \rightarrow 0^+, \beta \rightarrow t^-$ in (6.9) yields

$$\begin{aligned} & - \int_0^t \int_M h_q(W(s, m, q), (D^2 + \frac{\partial}{\partial t})W'(t - s, q, p)) dq(g) ds = \\ & = \lim_{\beta \rightarrow t^-} \int_M h_q(W(\beta, m, q), W'(t - \beta, q, p)) dq(g) - W'(t, m, p). \end{aligned} \quad (6.10)$$

$g' \in \text{comp}(g)$ implies

$$dq(g) = \alpha(q) dq(g') \quad (6.11)$$

$0 < c_1 \leq \alpha(q) \leq c_2$. We rewrite

$$\begin{aligned} & \lim_{\beta \rightarrow t^-} \int_M h_q(W(\beta, m, q), W'(t - \beta, q, p)) dq(g) = \\ & = \lim_{\beta \rightarrow t^-} \int_M h'(W(\beta, m, q), W'(t - \beta, q, p)) \alpha(q) (\alpha(q)^{-1} dq(g)) + \end{aligned}$$

$$\begin{aligned}
& + \lim_{\beta \rightarrow t^-} \int_M (h - h')_q(W(\beta, m, q), W'(t - \beta, q, p)) dq(g) = \\
& = \alpha(p) \cdot W(t, m, p) + \lim_{\beta \rightarrow t^-} \int_M (h - h')_q(W(\beta, m, q), W'(t - \beta, q, p)) dq(g)
\end{aligned}$$

and obtain

$$\begin{aligned}
& - \int_0^t \int_M h_q(W(s, m, q), (D^2 + \frac{\partial}{\partial t})W'(t - s, q, p)) dq(g) ds = \\
& = - \int_0^t \int_M h_q(W(s, m, q), (D^2 - D'^2)W'(t - s, q, p)) dq(g) ds = \\
& = \alpha(p)W(t, m, p) - W'(t, m, p) + \\
& + \lim_{\beta \rightarrow t^-} \int_M (h - h')_q(W(\beta, m, q), W'(t - \beta, p, q)) dq(g). \tag{6.12}
\end{aligned}$$

We see immediately that (6.12) expresses the operator equation

$$\begin{aligned}
& e^{-tD^2} - e^{-tD'^2} = - \int_0^t e^{-sD^2} (D^2 - D'^2) e^{-(t-s)D'^2} ds - \\
& - \int_M h'_p \left(\lim_{\beta \rightarrow t^-} \int_M (h - h')_q(W(\beta, m, q), W'(t - \beta, q, p)) dq(g), \cdot \right) dp(g') \tag{6.13}
\end{aligned}$$

in $L_2((M, E), h', g')$ at kernel level. We want to show that both terms on the right hand side of (6.13) are trace class operators with uniformly bounded trace norm on compact t -intervalls $[a_0, a_1]$, $a_0 > 0$, and we start with

$$\begin{aligned}
& \int_0^t e^{-sD^2} (D^2 - D'^2) e^{-(t-s)D'^2} ds = \\
& = \int_0^t e^{-sD^2} D(D - D') e^{-(t-s)D'^2} ds + \tag{6.14}
\end{aligned}$$

$$+ \int_0^t e^{-sD^2} (D - D') D' e^{-(t-s)D'^2} ds. \tag{6.15}$$

Write $D = \sum_{i=1}^n e_i \cdot \nabla_{e_i}$, $D' = \sum_{i=1}^n e'_i \cdot \nabla_{e'_i}$. Then $(D - D')\Phi = \sum_{i=1}^n e_i \cdot \nabla_{e_i} \Phi - e'_i \cdot \nabla_{e'_i} \Phi$. Consider $e \cdot \nabla_e - e' \cdot \nabla_{e'} = (e - e') \cdot \nabla_e + e' \cdot (\nabla_e - \nabla_{e'}) + e' \cdot (\nabla_{e'} - \nabla_{e'}) + e'(\cdot - \cdot) \nabla_{e'}$. Hence

$$(D - D')\Phi = (\eta_1 + \eta_2 + \eta_3 + \eta_4)\Phi,$$

where

$$\begin{aligned}
\eta_1 \Phi &= \sum_i (e_i - e'_i) \cdot \nabla_{e_i} \Phi , \\
\eta_2 \Phi &= \sum_i e'_i \cdot (\nabla_{e_i} - \nabla_{e'_i}) \Phi , \\
\eta_3 \Phi &= \sum_i e'_i \cdot (\nabla_{e'_i} - \nabla'_{e'_i}) \Phi , \\
\eta_4 \Phi &= \sum_i e'_i (\cdot - \cdot') \nabla'_{e'_i} \Phi .
\end{aligned}$$

We simply write η_ν instead η_ν^{op} and obtain

$$(6.14) + (6.15) = \int_0^t e^{-sD^2} D(\eta_1 + \eta_2 + \eta_3 + \eta_4) e^{-(t-s)D'^2} ds + \quad (6.16)$$

$$+ \int_0^t e^{-sD^2} (\eta_1 + \eta_2 + \eta_3 + \eta_4) D' e^{-(t-s)D'^2} ds. \quad (6.17)$$

We have to estimate

$$\int_0^t e^{-sD^2} D\eta_\nu e^{-(t-s)D'^2} ds \quad (6.18)$$

and

$$\int_0^t e^{-sD^2} \eta_\nu D' e^{-(t-s)D'^2} ds. \quad (6.19)$$

Decompose $\int_0^t = \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t$ which yields

$$\int_0^{\frac{t}{2}} e^{-sD^2} D\eta_\nu e^{-(t-s)D'^2} ds, \quad (I_{\nu,1})$$

$$\int_0^{\frac{t}{2}} e^{-sD^2} \eta_\nu D' e^{-(t-s)D'^2} ds, \quad (I_{\nu,2})$$

$$\int_{\frac{t}{2}}^t e^{-sD^2} D\eta_\nu e^{-(t-s)D'^2} ds, \quad (I_{\nu,3})$$

$$\int_{\frac{t}{2}}^t e^{-sD^2} \eta_\nu D' e^{-(t-s)D'^2} ds. \quad (I_{\nu,4})$$

$(I_{\nu,j})$ has the same structure as (I_j) in section 5.

But in distinction to section 5, not all $\eta_\nu = \eta_\nu^{op}$ are operators of order zero. Only η_3 is zero order operator, generated by an End E valued 1-form η_3 . For it we want to show and then to use

$$|\eta_3|_{1,r-1} < \infty, \quad (6.20)$$

where $|\cdot|_{1,r-1} = |\cdot|_{g,h,\nabla,1,r-1}$ or $|\cdot|_{g',h',\nabla',1,r-1}$ as we want. $\eta_1^{op}, \eta_2^{op}, \eta_4^{op}$ are first order operators. For them we want to show that their coefficients decrease sufficiently fast, i. e. have finite $|\cdot|_{1,r-1}$ -norm.

Altogether we have to estimate 16 integrands which split into even more.

We start with $\nu = 3$. (6.20) is an immediate consequence of (6.3), (6.4) and we are from an analytical point of view exactly in the situation of section 5. $(I_{3,1}) - (I_{3,4})$ can be estimated quite parallel to $(I_1) - (I_4)$ in section 5 and we are done. There remains the estimate of $(I_{\nu,j}), \nu \neq 3, j = 1, \dots, 4$. Start with $\nu = 1, j = 3$ write

$$\begin{aligned} & e^{-sD^2} D \eta_1 e^{-(t-s)D'^2} = \\ & = \left((D e^{-\frac{sD^2}{2}}) \circ f \right) \circ \left(f^{-1} e^{-\frac{sD^2}{2}} \eta_1 \right) \circ e^{-(t-s)D'^2}. \end{aligned} \quad (6.21)$$

(6.21) holds since $e^{-\frac{sD^2}{2}}$ is a smoothing operator. $e^{-(t-s)D'^2}$ is bounded in $[\frac{t}{2}, t]$ and we handle and estimate it as in section 5. $((D e^{-\frac{sD^2}{2}}) \circ f)$ is Hilbert–Schmidt if $f \in L_2$. There remains to show that for appropriate f

$$f^{-1} e^{-\frac{sD^2}{2}} \eta_1 \quad (6.22)$$

is Hilbert–Schmidt. Recall $r > n + 2, n \geq 2$, which implies $\frac{r}{2} > \frac{n}{2} + 1, r - 1 - n \geq \frac{r}{2} - \frac{n}{2}, r - 1 \geq \frac{r}{2}, 2 \geq i$. If we write in the sequel pointwise or Sobolev norms we should always write $|\Psi|_{g',h',m}, |\Psi|_{H^r(E,D')}, |\Psi|_{g',h',\nabla',2,\frac{r}{2}}, |g - g'|_{g',m}, |g - g'|_{g',1,r}$ etc. But we omit the reference to g', h, ∇', D, m' in the denotation for the sake of brevity. Moreover, as we already know, g, h, ∇, D generate equivalent norms.

$$\text{Now } (\eta_1 \Phi)(m) = \sum_{i=1}^n (e_i - e'_i) \cdot \nabla_{e_i} \Phi,$$

$$|\eta_1 \Phi|_m \leq \left(\sum_{i=1}^n |e_i - e'_i|_m^2 \right)^{\frac{1}{2}} \cdot |\nabla \Phi|_m \leq |g - g'|_m \cdot |\nabla \Phi|_m. \quad (6.23)$$

Similarly, for $\text{supp } \Phi$ compact, $|\Phi|_{L_2} = 1, |\Phi|_{H^{\frac{r}{2}}} \leq C_1, s > 0$,

$$\begin{aligned} & |\eta_1 \Phi|_{H^{\frac{r}{2}-1}, B_{2d+s}(m)} \leq \\ & \leq C_2 |g - g'|_{2,\frac{r}{2}, B_{2d+s}(m)} \leq \quad (\text{since } r > n + 2) \\ & \leq C_3 |g - g'|_{1,r-1, B_{2d+s}(m)} = \\ & = C_3 \text{vol}(B_{2d+s}(m)) \cdot \\ & \cdot \left(\frac{1}{\text{vol} B_{2d+s}(m)} |g - g'|_{1,r-1, B_{2d+s}(m)} \right), \end{aligned} \quad (6.24)$$

and then we proceed as in (5.31)–(5.40), i. e. we set

$$\begin{aligned} \tilde{\eta}_1(m) &:= C_4 \int_0^\infty e^{-\frac{9}{10} \frac{s^2}{4t}} \left(\frac{1}{\text{vol} B_{2d+s}(m)} |g - g'|_{1,r-1, B_{2d+s}(m)} \right) ds, \\ f_1(m) &:= (\tilde{\eta}_1(m))^{\frac{1}{2}}, \end{aligned}$$

and obtain finally

$$\begin{aligned} |\text{integral kernel } f_1^{-1} e^{-\frac{s}{2} D^2} \eta_1|_{L_2(M \times M)} &\leq C_5(t), \\ \left| \int_{\frac{t}{2}}^t e^{-s D^2} D \eta_1 e^{-(t-s) D'^2} ds \right|_1 &\leq C_6\left(\frac{t}{2}, t\right) \cdot \sqrt{t}. \end{aligned} \quad (6.25)$$

$(I_{1,3})$ is done. $(I_{1,1}), (I_{1,2}), (I_{1,4})$ can be handled parallel to $(I_1), (I_2), (I_4)$ of section 5. If at the "continuous" end appear additional second derivatives, we proceed as with (5.54) using the version of (5.37), i. e. (6.24).

Now it is completely clear that $(I_{\nu,j}), \nu = 2, 4, j = 1, \dots, 4$, are done if we have an estimate for η_2, η_4 as above, coming from our assumptions.

$$\begin{aligned} |\eta_2 \Phi|_m &= \left| \sum_i e'_i (\nabla_{e_i} - \nabla_{e'_i}) \Phi \right|_m = \left| \sum_i e'_i \nabla_{e_i - e'_i} \Phi \right|_m \leq \\ &\leq \left(\sum_i |e_i - e'_i|_m^2 \right)^{\frac{1}{2}} |\nabla \Phi|_m \leq |g - g'|_m |\nabla \Phi|_m \end{aligned}$$

Similarly for higher derivatives and we proceed as for η_1 .

There remains η_4 .

$$\begin{aligned} (\eta_4 \Phi)(m) &= \sum_i e'_i (\cdot - \cdot') \nabla'_{e'_i} \Phi = \sum_i (\cdot - \cdot') (e'_i \otimes \nabla'_{e'_i} \Phi) \\ |\eta_4 \Phi|_m &\leq |\cdot - \cdot'|_m \cdot |\nabla' \Phi| \end{aligned}$$

Using our assumption (Clm)

$$(\cdot - \cdot') \in W^{1,r}(\text{Hom}(TM \otimes E, E)),$$

we proceed as for the other η_ν .

Finally we have to show that the operator

$$\Phi \rightarrow \int_M h'_p \left(\lim_{\beta \rightarrow t^-} \int_M (h - h')_q (W(\beta, m, q), W'(t - \beta, q, p)) dq(g), \Phi(p) \right) dp(g') \quad (6.26)$$

is a product of Hilbert-Schmidt operators. The first step is to rewrite (6.26). For doing this, we apply the following facts

1. $\lim_{\beta \rightarrow t^-} \int h'_p (W'(t - \beta, q, p), \cdot) dp(g') = \delta(q), \Phi(p) \in C_c^\infty$
2. $W, W' \in C^\infty(\mathbf{R}_+ \times M \times M, E \boxtimes E)$
3. For $a, b \in (V, (\cdot, \cdot)_V), a', b' \in (V', (\cdot, \cdot)_{V'})$ holds
 $(a \otimes a', b \otimes b')_{V \otimes V'} = (a, b)_V \cdot (a', b')_{V'} = ((a \otimes a', b), b') = (a, (a', b \otimes b'))$
4. The principle of Tonelli and the Fubini theorem for absolutely integrable integrands.

Then we can rewrite (6.26) as

$$\Phi \rightarrow \int_M (h - h')_q (W(t, m, q), \Phi(q)) dq(g) \quad (6.27)$$

We decompose (6.27) as

$$\begin{aligned}
& \int_M (h - h')_q(W(t, m, q), \Phi(q)) dq(g) = \\
&= \int_M (h - h')_q \left(\int_M h_u(W(\frac{t}{2}, m, u), W(\frac{t}{2}, m, u)) du(g), \Phi(q) \right) dq(g) = \\
&= \int_M h_u \left(W(\frac{t}{2}, m, u), \int_M (h - h')_q(W(\frac{t}{2}, u, q), \Phi(q)) dq(g) \right) du(g) = \\
&= A_2(A_1\Phi),
\end{aligned} \tag{6.28}$$

where

$$(A_1\Phi)(u) = \int_M (h - h')_q(W(\frac{t}{2}, u, q), \Phi(q)) dq(g)$$

and

$$(A_2\Psi)(m) = \int_M h_u(W(\frac{t}{2}, m, u), \Psi(u)) du(g).$$

Next we want to write

$$A_2 \circ A_1 = (A_2 \circ f \cdot) \circ ((f^{-1} \cdot) \circ A_1), \tag{6.29}$$

f a scalar function s. t. $A_2 \circ f$ and $(f^{-1} \cdot) \circ A_1$ are Hilbert-Schmidt operators and we start with A_1 . Our procedure is as above. We estimate the integral norm of A_1 with respect to one variable and then we define f . Now

$$\begin{aligned}
& \left| (h - h')_q(W(t, u, q), \cdot) \right|_{L_2(h, dq(g))}^2 \leq \\
& \leq \left| h(W(t, u, q), (|h - h'|_{h, q} \cdot)) \right|_{L_2(h, dq(g))}.
\end{aligned} \tag{6.30}$$

This amounts as in section 5 to estimate

$$\begin{aligned}
& \sup_{\substack{\Phi \in C_c^\infty \\ |\Phi|_{L_2} = 1 \\ |\Phi|_{H^{\frac{r}{2}}} \leq C}} |N(\Phi)|^2,
\end{aligned}$$

where

$$\begin{aligned}
N(\Phi) &= \left| \langle \delta(u), e^{-tD^2}(|h - h'|_{h, q} \cdot \Phi) \rangle \right| = \\
&= \frac{1}{\sqrt{4\pi t}} \left| \int_{-\infty}^{+\infty} e^{-\frac{s^2}{4t}} e^{isD}(|h - h'|_h \cdot \Phi) ds \right|.
\end{aligned} \tag{6.31}$$

But now we proceed literally as is (5.19) – (5.35), replacing η by $|h - h'|$ and setting

$$\tilde{f} := C_{10} \cdot \int_0^\infty e^{-\frac{9}{16} \frac{s^2}{4} \frac{t}{2}} \left(\frac{1}{\text{vol} B_{2d+s}(u)} |h - h'|_{h, \nabla, r-1, B_{2d+s}(u)} \right) ds.$$

Then again (for $h \neq h'$) $\tilde{f}(u)$ is $\neq 0$ everywhere and 1-summable,

$$\int_M \left| (h - h')_q(W(\frac{t}{2}, u, q), \cdot) \right|_{h,q}^2 dq(g) \leq \tilde{f}(u)^2. \quad (6.32)$$

We set $f(u) := (\tilde{f}(u))^{\frac{1}{2}}$ and infer $f(u) \neq 0$ everywhere, $f \in L_2$ and

$$\begin{aligned} \int_M \int_M f(u)^{-2} \left| (h - h')_q(W(\frac{t}{2}, u, q), \cdot) \right|_{h,q}^2 dq(g) du(g) &\leq \\ &\leq \int_M \frac{1}{\tilde{f}(u)} \tilde{f}(u)^2 du(g) = \int_M \tilde{f}(u) du(g) \leq C_{11}(t) \cdot \sqrt{t}, \end{aligned} \quad (6.33)$$

where for $t \in [a_0, a_1], a_0 > 0$,

$$\sup_{t \in [a_0, a_1]} C_{11}(t) \cdot \sqrt{t} = C_{12}(a_0, a_1) < \infty. \quad (6.34)$$

Quite similarly as in (5.12) – (5.14),

$$\int_M \int_M f(m)^2 |W(\frac{t}{2}, m, u)|_h^2 du(g) dm(g) \leq C_{13}(t), \quad (6.35)$$

and finally, as in (5.39) – (5.40)

$$\begin{aligned} \left| \Phi \rightarrow \int_M h'_p \left(\lim_{\beta \rightarrow t^-} \int_M (h - h')_q(W(\beta, m, q), W'(t - \beta, q, p)) dq(g), \Phi(p) \right) dp(g') \right|_1 \\ \leq C_{14}(t), \end{aligned} \quad (6.36)$$

where for $[a_0, a_1], a_0 > 0$,

$$\sup_{t \in [a_0, a_1]} C_{14}(t) = C_{15}(a_0, a_1) < \infty, \quad (6.37)$$

this finishes the proof of 6.1. \square

Example. The simplest standard example is $E = (\Lambda^* T^* M, g_{\Lambda^*}, \nabla^{g_{\Lambda^*}})$ with Clifford multiplication

$$x \otimes \omega \in T_m M \otimes \Lambda^* T_m^* M \rightarrow X \cdot \omega := \omega_X \wedge \omega - i_X \omega,$$

where $\omega_X := g(\cdot, X)$. In this case, E as a vector bundle remains fixed but the Clifford module structure varies smoothly with $g, g' \in \text{comp}(g)$, i.e. (6.1), automatically implies (Clm), (6.2), (6.3), (6.4). It is well known that in this case $D = d + d^*$, $D^2 = (d + d^*)^2 =$ graded Laplace operator Δ . Hence we obtain

Corollary 6.7. *Assume (M^n, g) with $(I), (B_k), k \geq r > n + 2, g' \in \mathcal{M}(I, B_k), g' \in \text{comp}(g) \subset \mathcal{M}^{1,r}(I, B_k)$. Then*

$$e^{-t\Delta} - e^{-t\Delta'}$$

is of trace class and the trace norm is uniformly bounded on compact t -intervals $[a_0, a_1], a_0 > 0$.

\square

Remark. We are also able to prove 6.7 directly without reference to 6.1. For this we write $\Delta' = \Delta + \eta$, calculate and estimate η (which is very easy), apply Duhamel's principle and proceed as before. \square

We need in section 7 the theorem analogous to 5.3 for the case of additional variation of the metrics.

Theorem 6.8. *Suppose the hypotheses of 6.1, replacing $r > n + 2$ by $r > n + 3$. Then*

$$De^{-tD^2} - D'e^{-tD'^2}$$

is of trace class and the trace norm is uniformly bounded on compact t -intervals $[a_0, a_1]$, $a_0 > 0$.

Proof. The proof is a simple combination of the proofs of 5.3 and 6.1. \square

7 Relative index theory

We now assume that E is endowed with an involution $\tau : E \rightarrow E$, s. t.

$$\tau^2 = 1, \tau^* = \tau, \tag{7.1}$$

$$[\tau, X]_+ = 0 \text{ for } X \in TM, \tag{7.2}$$

$$[\nabla, \tau] = 0. \tag{7.3}$$

Then $L_2(M, E) = L_2(M, E^+) \oplus L_2(M, E^-)$,

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$$

and $D^- = (D^+)^*$. If M^n is compact then as usual

$$\text{ind } D := \text{ind } D^+ := \dim \ker D^+ - \dim \ker D^- \equiv \text{tr } (\tau e^{-tD^2}), \tag{7.4}$$

where we understand τ as

$$\tau = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} : L_2(E^*) \oplus L_2(E^-) \rightarrow L_2(E^+) \oplus L_2(E^-).$$

For open M^n $\text{ind } D$ in general is not defined since $\dim \ker D^+$, $\dim \ker D^-$ are not of trace class.

The appropriate approach on open manifolds is relative index theory for pairs of operators D, D' . If $e^{-tD^2} - e^{-tD'^2}$ is of trace class then

$$\text{ind}(D, D') := \text{tr } (\tau(e^{-tD^2} - e^{-tD'^2})) \tag{7.5}$$

makes sense, but at the first glance (7.5) should depend on t .

Proposition 7.1. Suppose $e^{-tD^2} - e^{-tD'^2}$ and $De^{-tD^2} - D'e^{-tD'^2}$ of trace class for all $t > 0$ and $|De^{-tD^2} - D'e^{-tD'^2}|_1$ uniformly bounded on any compact t -intervall $[a_0, a_1], a_0 > 0$. Then $\text{tr}(\tau(e^{-tD^2} - e^{-tD'^2}))$ is independent of t .

See [1] for a proof. \square

Corollary 7.2. Assume the hypotheses of 7.1. Then $\text{ind}(D, D')$ is independent of t and hence well defined.

Corollary 7.3. Assume the hypotheses of 7.1 and $\inf \sigma_e(D^2) > 0$. Then $\text{ind } D, \text{ind } D'$ are well defined and

$$\text{tr}(\tau(e^{-tD^2} - e^{-tD'^2})) = \text{ind } D - \text{ind } D'. \quad (7.6)$$

Proof. From our assumptions, $\sigma_e(D^2) = \sigma_e(D'^2)$. $\inf \sigma_e(D^2) > 0$ immediately implies $\dim \ker D^+, \dim \ker D^- < \infty$. According to (0.7) of [11], there exists a constant $c > 0$ s. t.

$$\text{tr}(\tau(e^{-tD^2} - e^{-tD'^2})) = \text{ind } D - \text{ind } D' + O(e^{-ct}). \quad (7.7)$$

Performing $\lim_{t \rightarrow \infty}$ in (7.7) and using 7.2, we obtain (7.6). \square

Assume now the hypotheses of 5.1. Then we have asymptotic expansions

$$\text{tr}(\tau W(t, m, m)) \underset{t \rightarrow 0^+}{\sim} t^{-\frac{n}{2}} b_{-\frac{n}{2}}(D, m) + \dots + b_0(D, m) + b_{\frac{1}{2}}(D, m)t^{\frac{1}{2}} + \dots \quad (7.8)$$

$$\text{tr}(\tau W'(t, m, m)) \underset{t \rightarrow 0^+}{\sim} t^{-\frac{n}{2}} b_{-\frac{n}{2}}(D', m) + \dots + b_0(D', m) + b_{\frac{1}{2}}(D', m)t^{\frac{1}{2}} + \dots \quad (7.9)$$

We show in the next section that

$$b_i(D, m) - b_i(D', m) \in L_1, \quad -\frac{n}{2} \leq i \leq 1. \quad (7.10)$$

Define

$$\text{ind}_{\text{top}}(D, D') := \int_M (b_0(D, m) - b_0(D', m)) dm.$$

According to (7.10), $\text{ind}_{\text{top}}(D, D')$ is well defined.

Theorem 7.4. Suppose the hypotheses of 5.3. Then

$$\text{ind}(D, D') = \text{ind}_{\text{top}}(D, D')$$

If additionally $\inf \sigma_e(D^2) > 0$ then

$$\text{ind}_{\text{top}}(D, D') = \text{ind } D - \text{ind } D'.$$

Proof. This follows immediately from 7.2, 7.3, (7.8), (7.9) and the fact that the L_2 -trace of a trace class integral operator is given by the integral of the kernel on the diagonal (after forming pointwise traces). \square

If we admit variation of g too as in section 6, then the heat kernel of $e^{-tD'^2}$ in $L_2((M, E), g, h)$ is given by $\alpha(m)^{\frac{1}{2}}W'(t, m, p)\alpha(p)^{-\frac{1}{2}}$. But on the diagonal the α 's cancel out and the asymptotic expansion of $W'(t, m, m)$ with respect to $L_2(g)$ is the same as with respect to $L_2(g')$. We obtain for $W(t, m, m)$ or $W'(t, m, m)$ heat kernel coefficients $b_i(D(g, h, \nabla), m)$ or $b_i(D(g', h', \nabla'), m)$, respectively. We show in the next section that under the hypotheses of 6.1

$$b_i(D(g, h, \nabla), m) - b_i(D(g', h', \nabla'), m) \in L_1, \quad -\frac{n}{2} \leq i \leq 1. \quad (7.11)$$

Theorem 7.5 *Suppose the hypotheses of 6.8. Then*

$$\text{ind}(D, D') = \text{ind}_{\text{top}}(D, D').$$

If additionally $\inf \sigma_e(D^2) > 0$, then

$$\text{ind}_{\text{top}}(D, D') = \text{ind } D - \text{ind } D'.$$

The proof runs through literally as that of 7.4. \square

Deeper results on the relative index using scattering theory will be established in a forthcoming paper.

8 Relative ζ -functions, determinants and torsion

We start with a pair D, D' assuming the hypotheses of 5.1. Then we have the asymptotic expansion

$$\text{tr } W(t, m, m) \underset{t \rightarrow 0^+}{\sim} t^{-\frac{n}{2}} b_{-\frac{n}{2}}(m) + t^{-\frac{n}{2}+1} b_{-\frac{n}{2}+1} + \dots \quad (8.1)$$

and analogously for $\text{tr } W'(t, m, m)$ with

$$b_{-\frac{n}{2}+l}(m) = b_{-\frac{n}{2}+l}(D(g, h, \nabla), m), \quad b'_{-\frac{n}{2}+l}(m) = b_{-\frac{n}{2}+l}(D(g, h, \nabla'), m).$$

The heat kernel coefficients have for $l \geq 1$ a representation

$$b_{-\frac{n}{2}+l} = \sum_{k=1}^l \sum_{q=0}^l \sum_{i_1, \dots, i_k \geq 0} \nabla^{i_1} R^g \dots \nabla^{i_q} R^g \text{tr} (\nabla^{i_{q+1}} R^E \dots \nabla^{i_k} R^E) C^{i_1, \dots, i_k}, \quad (8.2)$$

where C^{i_1, \dots, i_k} stands for a contraction with respect to g , i.e. it is built up by linear combination of products of the g^{ij} .

Lemma 8.1. $b_{-\frac{n}{2}+l} - b'_{-\frac{n}{2}+i} \in L_1(M, g)$, $0 \leq l \leq \frac{n+3}{2}$.

Proof. Forming the difference $b_{-\frac{n}{2}+l} - b'_{-\frac{n}{2}+l}$, we obtain a sum of terms of the kind

$$\nabla^{i_1} R^g \dots \nabla^{i_q} R^g \operatorname{tr} [\nabla^{i_{q+1}} R^E \dots \nabla^{i_k} R^E - \nabla^{i_{q+1}} R'^E \dots \nabla^{i_k} R'^E]. \quad (8.3)$$

g is here fixed. The highest derivative of R^q with respect to ∇^g occurs if $q = k, i_1 = \dots = i_{q-1} = 0$. Then we have

$$(\nabla^g)^{2l-2k}. \quad (8.4)$$

By assumption, we have bounded geometry of order $\geq r > n + 2$, i. e. of order $\geq n + 3$. Hence $(\nabla^g)^i R^g$ is bounded for $i \leq n + 1$. To obtain bounded $\nabla^j R^g$ -coefficients of $[\dots]$ in (8.3), we must assume

$$2l - 2 \leq n + 1, \quad l \leq \frac{n + 3}{2}. \quad (8.5)$$

Similarly we see that the highest occurring derivatives of R^E, R'^E in $[\dots]$ are of order $2l - 2$. The corresponding expression

$$R^E \nabla^{2l-2} R^E - R'^E \nabla^{2l-2} R'^E = (R^E - R'^E)(\nabla^{2l-2} R^E) + R'^E(\nabla^{2l-2} R^E - \nabla^{2l-2} R'^E). \quad (8.6)$$

We want to apply the module structure theorem. $\nabla - \nabla' \in \Omega^{1,1,r}(\mathcal{G}_E^{Cl}, \nabla) = \Omega^{1,1,r}(\mathcal{G}_E^{Cl}, \nabla')$ implies $R^E - R'^E \in \Omega^{2,1,r-1}$. We can apply the module structure theorem (and conclude that all norm products of derivatives of order $\leq 2l - 2$ are absolutely integrable) if $2l - 2 \leq r - 1$, $2l - 2 \leq n + 1$, $l \leq \frac{n+3}{2}$. Hence, (8.5) $\in L_1$ since R^E, R'^E bounded. It is now a very simple combinatorial matter to write $[\dots]$ in (8.3) as a sum of terms each of them is a product of differences $(\nabla^i R^E - \nabla^i R'^E)$ with bounded functions $\nabla^j R^E, \nabla^{j'} R'^E$. Remember $\nabla, \nabla' \in \mathcal{C}_E(B_k)$. This proves 8.1. \square

Lemma 8.2. *There is an expansion*

$$\operatorname{tr}(e^{-tD^2} - e^{-tD'^2}) = t^{-\frac{n}{2}} a_{-\frac{n}{2}} + \dots + t^{-\frac{n}{2} + [\frac{n+3}{2}]} a_{-\frac{n}{2} + [\frac{n+3}{2}]} + O(t^{-\frac{n}{2} + [\frac{n+3}{2}] + 1}). \quad (8.7)$$

Proof. Set

$$a_{-\frac{n}{2}+i} = \int (b_{-\frac{n}{2}+i}(m) - b'_{-\frac{n}{2}+i}(m)) dm \quad (8.8)$$

and use

$$\operatorname{tr} W(t, m, m) = t^{-\frac{n}{2}} b_{-\frac{n}{2}} + \dots + t^{-\frac{n}{2} + [\frac{n+3}{2}]} b_{-\frac{n}{2} + [\frac{n+3}{2}]} + O(m, t^{-\frac{n}{2} + [\frac{n+3}{2}] + 1}), \quad (8.9)$$

$$\operatorname{tr} W'(t, m, m) = t^{-\frac{n}{2}} b'_{-\frac{n}{2}} + \dots + O'(m, t^{-\frac{n}{2} + [\frac{n+3}{2}] + 1}) \quad (8.10)$$

$$\operatorname{tr}(e^{-tD^2} - e^{-tD'^2}) = \int (\operatorname{tr} W(t, m, m) - \operatorname{tr} W'(t, m, m)) dm.$$

The only critical point is

$$\int_M O(m, t^{-\frac{n}{2} + [\frac{n+3}{2}] + 1}) - O'(m, t^{-\frac{n}{2} + [\frac{n+3}{2}] + 1}) dm = O(t^{-\frac{n}{2} + [\frac{n+3}{2}] + 1}). \quad (8.11)$$

(8.11) requires a very careful investigation of the concrete representatives for $O(m, t^{-\frac{n}{2} + [\frac{n+3}{2}]})$. We did this step by step, following [9], p. 21/22, 50–51. Very roughly spoken, the m -dependence of $O(m, \cdot)$ is given by the parametrix construction, i. e. by differences of corresponding derivatives of the $\Gamma_{i\alpha}^\beta, \Gamma_{i\alpha}'^\beta$, which are integrable by assumption. \square

Definition. Assume the hypotheses of 5.1. Set

$$\zeta_1(s, D, D') := \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \text{tr}(e^{-tD^2} - e^{-tD'^2}) dt. \quad (8.12)$$

Using 8.7,

$$\int_0^1 t^{s-1} t^{-\frac{n}{2} + [\frac{n+3}{2}]} dt = \frac{1}{s - \frac{n}{2} + [\frac{n+3}{2}]}, \quad (8.13)$$

$$\frac{1}{\Gamma(s)} \int_0^1 t^{s-1} O(t^{-\frac{n}{2} + [\frac{n+3}{2}] + 1}) dt \text{ holomorphic for } \text{Re}(s) + (-\frac{n}{2}) + [\frac{n+3}{2}] + 1 > 0 \quad (8.14)$$

and $[\frac{n+3}{2}] \geq \frac{n}{2} + 1$, we obtain a function meromorphic in $\text{Re}(s) > -1$, holomorphic in $s = 0$ with simple poles at $s = \frac{n}{2} - l$, $l \leq [\frac{n+3}{2}]$. Assume additionally $\inf \sigma_e(D^2) > 0$ and set $h = \dim \ker D^2 - \dim \ker D'^2$. Then

$$\text{tr}(e^{-tD^2} - e^{-tD'^2}) = \dim \ker D^2 - \dim \ker D'^2 + O(e^{-ct}) = h + O(e^{-ct}) \text{ for } t \rightarrow \infty, \quad c > 0. \quad (8.15)$$

Define for $\text{Re}(s) < 0$

$$\zeta_2(s, D^2, D'^2) := \frac{1}{\Gamma(s)} \int_1^\infty t^{s-1} \text{tr}(e^{-tD^2} - e^{-tD'^2}) dt = \frac{1}{\Gamma(s)} \int_1^\infty t^{s-1} (h + O(e^{-ct})) dt.$$

$\zeta_2(s, D^2, D'^2)$ is holomorphic in $\text{Re}(s) < 0$ and admits a meromorphic extension to \mathbf{C} which is holomorphic in $s = 0$.

Define

$$\zeta(s, D^2, D'^2) := \zeta_1(s, D^2, D'^2) + \zeta_2(s, D^2, D'^2).$$

We proved the following

Theorem 8.3. Suppose the hypotheses of 5.1 and additionally $\inf \sigma_e(D^2) > 0$. Then

$$\zeta(s, D^2, D'^2) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{tr}(e^{-tD^2} - e^{-tD'^2}) ds$$

is after meromorphic extension well defined in $\text{Re}(s) > -1$ and holomorphic in $s = 0$. \square

Definition. Suppose the hypotheses of 8.3. Then

$$\det(D^2, D'^2) := e^{-\zeta'(0, D^2, D'^2)}$$

is well defined and is called the relative determinant of D^2/D'^2 .

Remark. Fix $g, h, \nabla_0, D_0 = D(g, h, \nabla_0)$. Then we defined for any $D(g, h, \nabla), \nabla \in \text{comp}(\nabla_0) \cap \mathcal{C}_E(B_k) \subset \mathcal{C}_E^{1,r}(B_k)$ the relative determinant $\det(D^2, D'^2)$. \square

If we suppose the hypotheses of 6.1 then we can repeat the preceding considerations and estimates word by word.

Theorem 8.4. *Suppose the hypotheses of 6.1. and $\inf \sigma_e(D^2) > 0$. Then*

$$\zeta(s, D^2, D'^2) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{tr}(e^{-tD^2} - e^{-tD'^2}) dt$$

is after meromorphic extension well defined in $\text{Re}(s) > -1$ and holomorphic in $s = 0$. Hence the relative determinant

$$\det(D^2, D'^2) := e^{-\zeta'(0, D^2, D'^2)}$$

is well defined. \square

Corollary 8.5. *Suppose (M^n, g) with $(I), (B_k), k \geq r > n + 2, g' \in \text{comp}(g) \cap \mathcal{M}(I, B_k) \subset \mathcal{M}^{1,r}(I, B_k)$, and additionally $\inf \sigma_e(\Delta_q) > 0, q = 0, \dots, n$. Then the relative analytic torsion $\tau(M, g, g')$,*

$$\log \tau(M, g, g') := \frac{1}{2} \sum_{q=0}^n (-1)^q \frac{d}{ds} \zeta(s, \Delta_q, \Delta'_q) \Big|_{s=0}$$

is well defined. \square

In a forthcoming paper we drop considerably the assumption $\inf \sigma_e(\cdot) > 0$ and discuss further applications.

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